Fodder-mix-problem

(exercise after OHSE, D. (2004/2005): Mathematik für Wirtschaftswissenschaftler. Bd. I u. II, München: Vahlen)

Three different types of fodder (F_1, F_2, F_3) are designated to feed chicken on a farm.

These three types shall be mixed so that the feed contains

at least 80 units carbohydrates (C),

at least 120 units protein and (P)

at most 60 units fat (F).

The following table shows the price for one quantity unit of each fodder (F_1, F_2, F_3) and which amount of each nutrient one quantity unit contains.

	C	P	F	Price (€)
F_1	2	3	1	60
F_2	3	1	½	45
F_3	1	2	1	36

Which amount of each fodder (F_1, F_2, F_3) is required to get the lowest total cost?

The mathematical modelling of this exercise leads to this linear optimization problem.

$$\min \left\{ 60x_1 + 45x_2 + 36x_3 \;\middle|\; \begin{array}{l} 2x_1 + 3x_2 + x_3 \geq \;\; 80 \\ 3x_1 + x_2 + 2x_3 \geq 120,\; x_1,\, x_2,\, x_3 \geq 0 \\ x_1 + \frac{1}{2}x_2 + x_3 \leq \;\; 60 \end{array} \right\}$$

1. Set theoretic and arithmetic tools

1.1 Elements of set-theory

a) **Definition:**
A **set** is the bundling of objects. (To create a new object).

Symbolism:

1) Elements of a set: small latin letters

   A set: capital latin letters

2) $\mathbf{M} = \{\mathbf{x} \mid \mathbf{H(x)}\}$ The set is defined by the elements which comply with the condition or characteristic H (x)

3) $\mathbf{M} = \{\mathbf{x_1}, \mathbf{x_2}, ..., \mathbf{x_n}\}$ Enumeration of elements

4) $\mathbf{x_k} \in \mathbf{M}$ $\mathbf{x_k}$ is an element of M (k – Index).

5) $\exists$ Existence-Operator: It exists an element with...

   $\forall$ All-Operator: For all elements we assume...

b) Special sets

$\varnothing \quad \left(:= \{ \mathbf{x} \mid \mathbf{x} \neq \mathbf{x} \} \right)$	Empty set
$\mathbf{N} = \{ \mathbf{0, 1, 2, \ldots} \}$	Set of natural numbers
$\mathbf{Z} = \{ \mathbf{0}, +\mathbf{1}, -\mathbf{1}, +\mathbf{2}, -\mathbf{2}, \ldots \}$	Set of integer numbers
Q	Set of rational numbers (finite and recurring decimals)
R	Set of real numbers

Intervals play a key role as subsets of real numbers.

For real numbers a and b

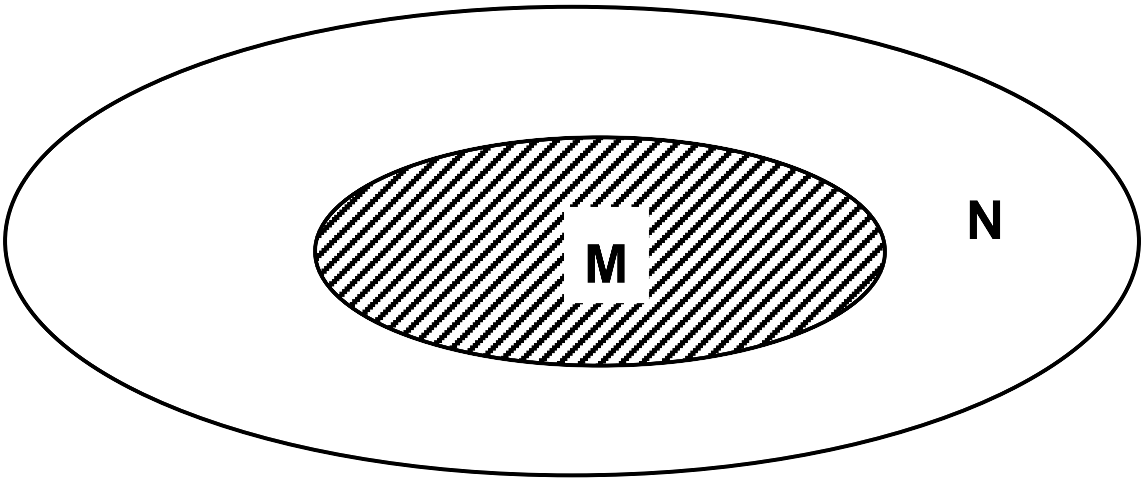
$(\mathbf{a},\mathbf{b}) \;:= \{ \mathbf{x} \in \mathbf{R} \mid \mathbf{a} < \mathbf{x} < \mathbf{b} \}$ describes an open interval,

$[\mathbf{a},\mathbf{b}] \;:= \{ \mathbf{x} \in \mathbf{R} \mid \mathbf{a} \leq \mathbf{x} \leq \mathbf{b} \}$ describes a closed interval.

c) Relations between sets

$\mathbf{M} \subset \mathbf{N}$ (M is **subset** of N) if and only if (abbreviation: iff.) each element of M is also an element of N.

VENN diagram:



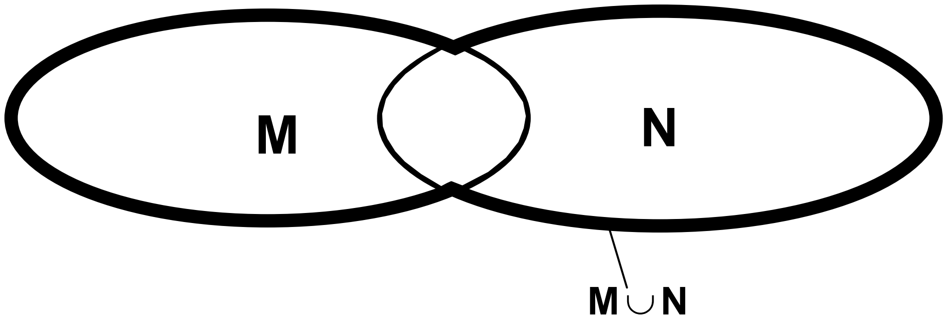
For each set M it is: $\emptyset \subset \mathbf{M}$.

$\mathbf{M} = \mathbf{N}$ iff. $\mathbf{M} \subset \mathbf{N}$ and $\mathbf{N} \subset \mathbf{M}$

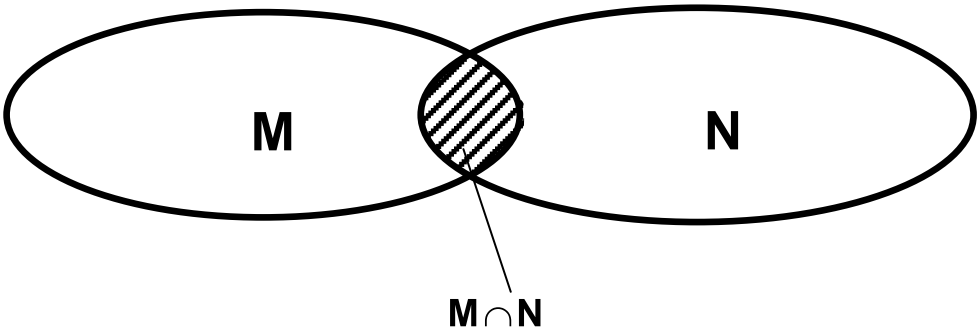
d) Operations with sets

(1) **Union** $\mathbf{M \cup N := \{ a \mid a \in M \ \underline{OR} \ a \in N \}}$

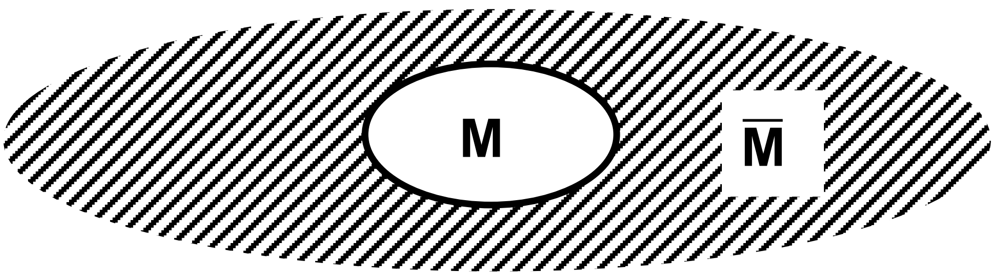
VENN diagram:



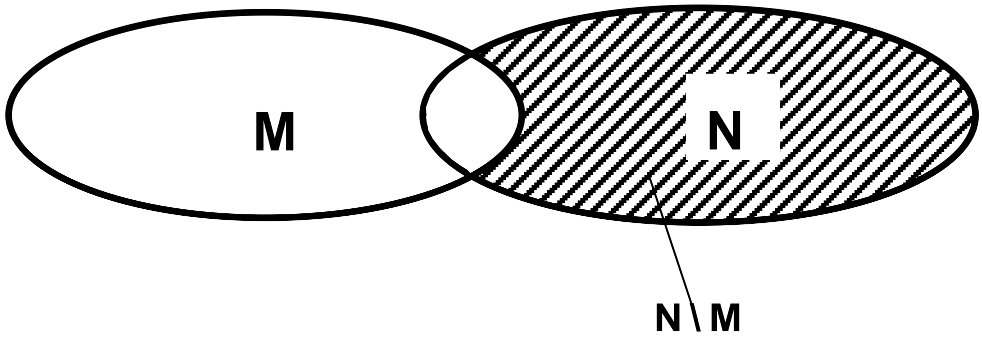
(2) **Intersection** $\mathbf{M \cap N := \{ a \mid a \in M \ \underline{AND} \ a \in N \}}$



(3) **Complement** $\mathbf{\overline{M} := \{ a \mid a \notin M \}}$



(4) **Set difference** $\mathbf{N \setminus M := \{a \mid a \in N \text{ and } a \notin M\}}$



(5) **Power set** $\mathbf{2^M}$ **of M** (Set of all sets of M)

$\mathbf{2^M := \{N \mid N \subset M\}}$

Example: $\mathbf{\{a, b\}}$

$\mathbf{2^{\{a,b\}} = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}}$

(6) **Ordered pair**

(a, b) ordered double-elemented pair

(a, b) = (c, d)	iff.	a = c	and	b = d
(f, g) = (3, -7)	iff.	f = 3		g = -7

(7) **Cartesian Product** (Cross product)

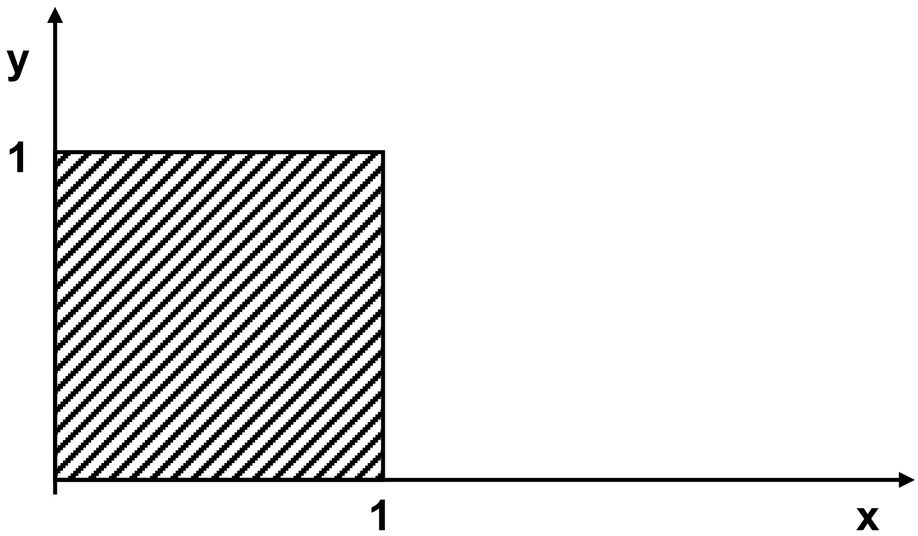
$$\mathbf{M \times N := \{(a,b) \mid a \in M \text{ and } b \in N\}}$$

Example:

1) $\mathbf{M = \{1,2\}, N = \{x,y\}, M \times N = \{(1,x), (1,y), (2,x), (2,y)\}}$

2) $\mathbf{X = \{x \mid 0 \le x \le 1\}}$

   $\mathbf{Y = \{y \mid 0 \le y \le 1\}}$



3) $\mathbf{R \times R = R^2 := \{(x,y) \mid x \in R, y \in R\}}$

   $\mathbf{R^3 := R \times R \times R}$

(8) f is called **mapping** of M into N iff. $\mathbf{f \subset M \times N}$.
Think about examples concerning the cross products mentioned above.

(9) **Inverse mapping $\mathbf{f^{-1}}$ of f** (also: Inverse of f)

$$\mathbf{f^{-1} := \{(a,b) \mid (b, a) \in f\}}$$

(10) **Function**

Definition:

A mapping f is called **function** or definite mapping of M in N iff.

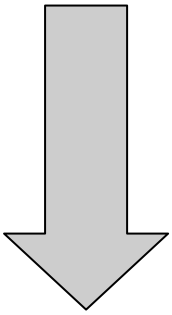
there is for every $\mathbf{a \in M}$ exactly one element $\mathbf{b \in N}$ with $\mathbf{(a,b) \in f}$.

Another writing for $\mathbf{(a,b) \in f}$ is $\mathbf{b = f(a)}$

Domain: $\mathbf{M}$

Range: $\mathbf{\{b \in N \mid \exists\, a \in M \text{ with } (a,b) \in f\}}$

Different stages of generalization:

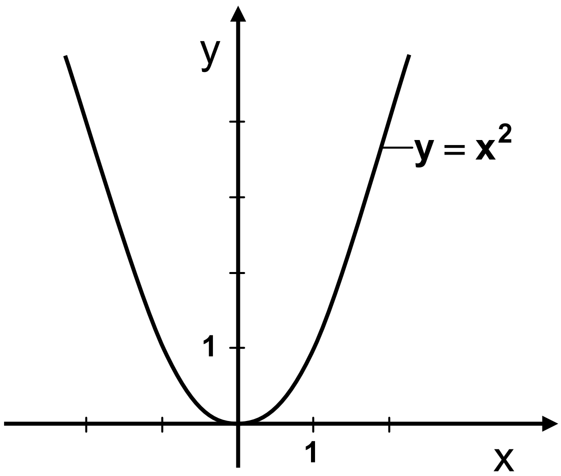


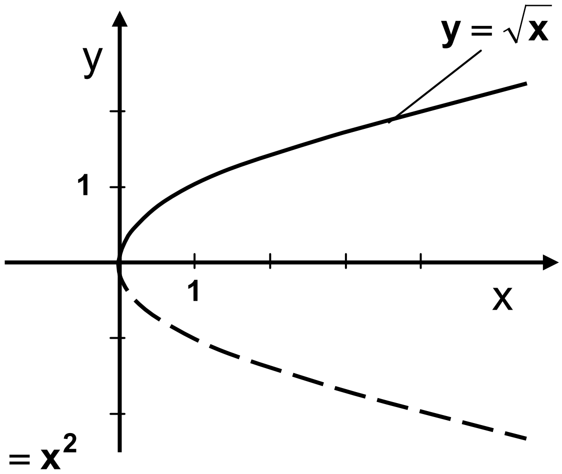
$\mathbf{y = x^2 + 3\ x - 2}$ fixed parabola at $(-\frac{3}{2}, -\frac{17}{4})$

$\mathbf{y = x^2 + p\ x + q}$ parabola with the parameters p and q

$\mathbf{y = f(x)}$ function

Example: $\mathbf{y = x^2}$





The existing inverse mapping of $\mathbf{y = x^2}$

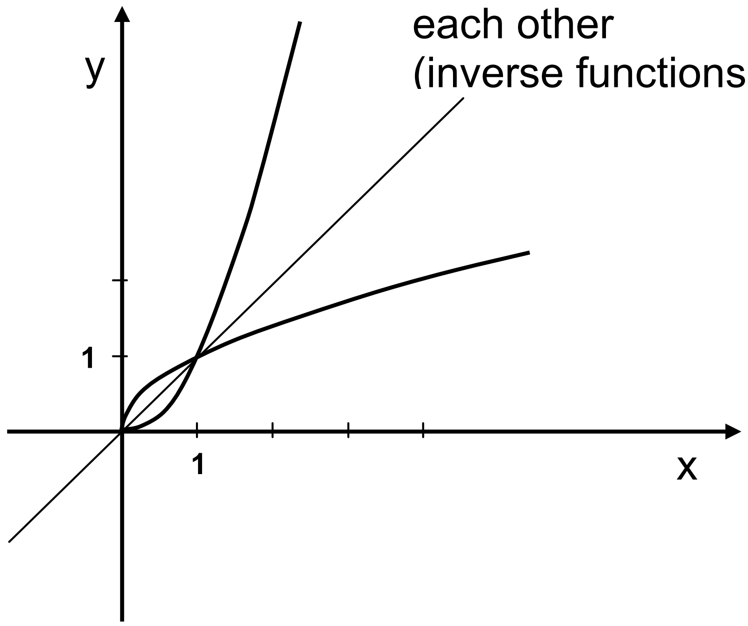
is not a function.

Definition:

f is called **one-to-one mapping** iff. $\mathbf{f}$ **and** $\mathbf{f^{-1}}$ are functions.

Example: $y = x^2$,for $x \geq 0$ ⟷ $y = \sqrt{x}$,for $x \geq 0$

Functions which are inverse to each other (inverse functions)



1.2 The sigma-notation

In reference to the fodder-mix-problem:

indexed variable:

x_i - amount of fodder type F_i

p_i - Price per quantity unit ($p_1 = 60,\ p_2 = 45,\ p_3 = 36$)

Solution: $x_1 = 0,\ x_2 = 8,\ x_3 = 56$

OF: 2376

OF: $60x_1 + 45x_2 + 36x_3 = p_1x_1 + p_2x_2 + p_3x_3 = \sum_{i=1}^{3} p_ix_i$

in general: $\sum_{i=k}^{n} a_i = a_k + a_{k+1} + \ldots + a_n$

Summation variable a_i

(A term, which mostly includes i, but not necessarily.)

Index of summation: i

lower and upper bound of summation: k, n

Examples: $\sum_{i=1}^{3} i^3 = 1^3 + 2^3 + 3^3 = 1 + 8 + 27 = 36$

$$\sum_{i=-2}^{4} 3 = 3 + 3 + 3 + 3 + 3 + 3 + 3 = 7 \cdot 3 = 21$$

Double sums, multiple sums

It is possible that the summation variable itself is a sum.

Example:

$$\sum_{i=1}^{2}\left(\sum_{j=-1}^{1} 2^{i}\cdot(3j+1)\right)=\sum_{j=-1}^{1} 2^{1}\ (3j+1)+\sum_{j=-1}^{1} 2^{2}\ (3j+1)$$

$$\begin{aligned}&= 2\ (-3+1) \quad + \ 2\ (3\cdot 0+1)+2\ (3+1)\\ &\quad + 2^{2}(-3+1)+4\ (3\cdot\ 0+1)\ +\ 4\ (3+1)\end{aligned}$$

$$\begin{aligned}&= -4 \quad + \quad 2 \quad + \quad 8\\ &\quad -8 \quad + \quad 4 \quad + \quad 16 = 18\\ &= \sum_{j=-1}^{1}\ \sum_{i=1}^{2} 2^{i}\cdot(3j+1)\end{aligned}$$

The product sign

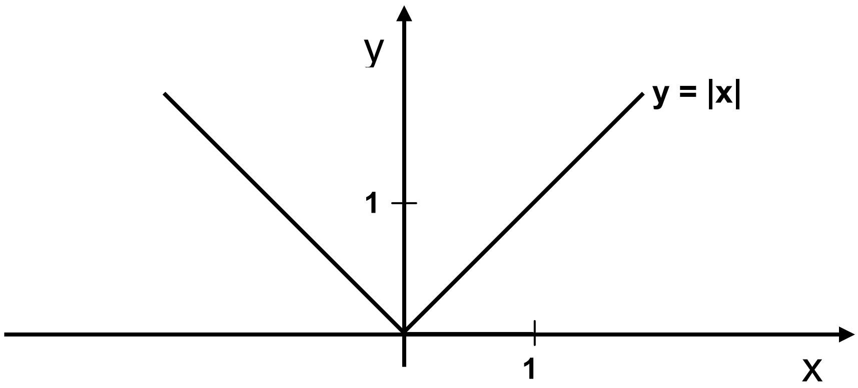
$\prod$ (Is used analogically)

Example:

$$n\,! := \ 1\cdot 2\cdot 3\cdot\ ...\ \cdot n=\prod_{i=1}^{n} i$$

1.3 The absolute value of a real number

Definition: $|\mathbf{x}| := \begin{cases} \mathbf{x}, \text{ if } \mathbf{x} \geq 0 \\ -\mathbf{x}, \text{ if } \mathbf{x} < 0 \end{cases}$ **absolute value** of $\mathbf{x} \in \mathbf{R}$



Conclusions:

- $|\mathbf{x}| = \max \{\mathbf{x}, -\mathbf{x}\}$
- $\mathbf{x} \leq |\mathbf{x}|, \quad -\mathbf{x} \leq |\mathbf{x}|, \quad |-\mathbf{x}| = |\mathbf{x}|$
- Triangle inequality: $\mathbf{x}, \mathbf{y} \in \mathbf{R}$

  $|\mathbf{x}+\mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$

  Generalization:

  $|\mathbf{x}_1 + \ldots + \mathbf{x}_n| \leq |\mathbf{x}_1| + \ldots + |\mathbf{x}_n|$ (principle of complete induction)

Inequalities with absolute values

(1) $|x| \le a \leftrightarrow -a \le x \le a \leftrightarrow x \in [-a, a]$

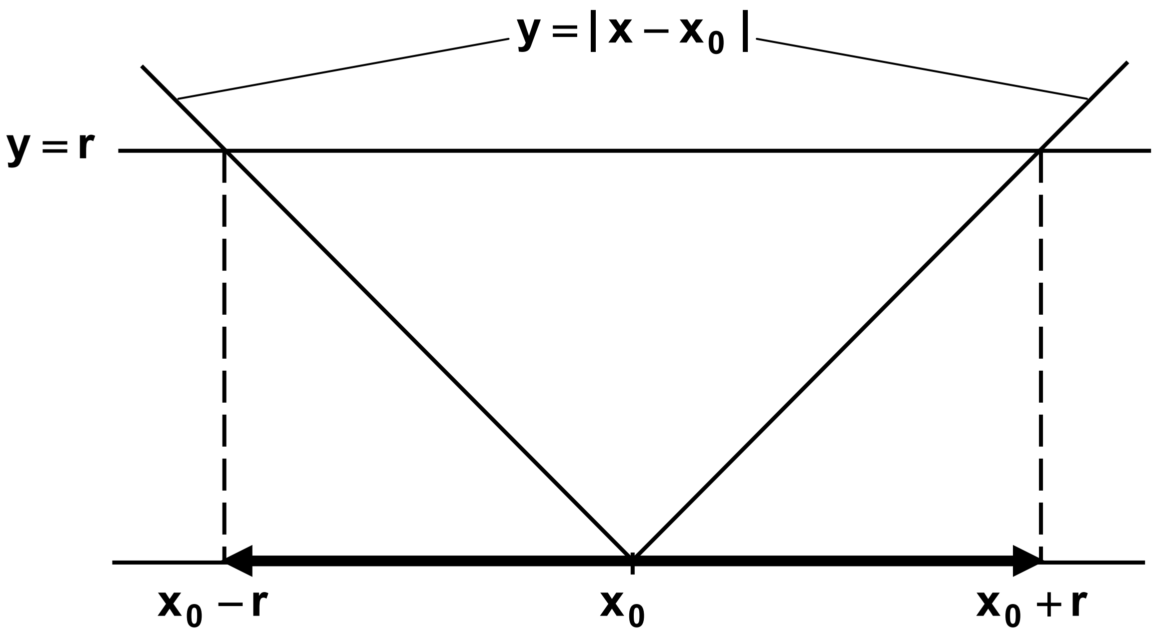
closed interval

(2) $r > 0, \quad \{x \in R \mid |x - x_0| < r\}$

$= \{x \in R \mid x \ge x_0, x < x_0 + r\} \cup \{x \in R \mid x < x_0, x > x_0 - r\}$

$= \{x \in R \mid -r < x - x_0 < r\} = \{x \in R \mid x_0 - r < x < x_0 + r\} = (x_0 - r, x_0 + r)$

open interval



1.4 Limit of a function

Fundamental idea in analysis

Use: Derivative of a function; elasticity

$\lim\limits_{x \to a} f(x) = b$

means: f (x) is arbitrarily close to b for every x, which is close enough to a.

What means „close to“? $|x - a|$, $|f(x) - b|$ → small,

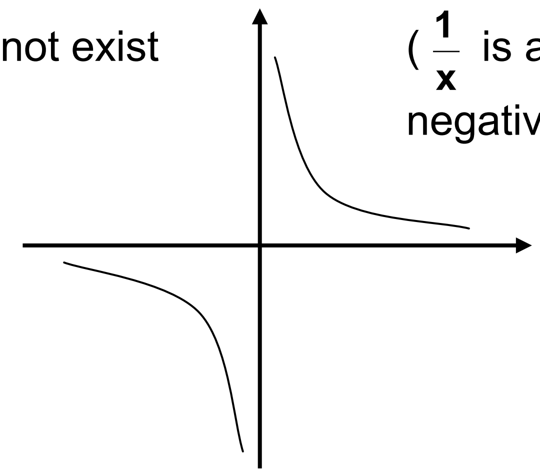
so

$\lim\limits_{x \to a} f(x) = b$ iff. $\forall \varepsilon > 0 \quad \exists \delta > 0 : |f(x) - b| < \varepsilon$ if $|x - a| < \delta$.

Examples:

$\lim\limits_{x \to 3} 2x = 6,$

$\lim\limits_{x \to 0} \frac{1}{x}$ does not exist ($\frac{1}{x}$ is a high positive or a high negative number)



2.1 Functions of a real variable

Definition: Be $\mathbf{M_1}$ and $\mathbf{M_2}$ two sets of real numbers. The assignment instruction f, which allocates any $\mathbf{x} \in \mathbf{M}_1$ exactly one element $\mathbf{y} \in \mathbf{M_2}$, is called a function.

$\mathbf{f} = \left\{ (\mathbf{x}, \mathbf{y}) \in \mathbf{R}^2 \;\middle|\; \mathbf{x} \in \mathbf{M_1}, \mathbf{y} = \mathbf{f(x)} \in \mathbf{M_2} \right\}$ as definite mapping.

Symbolism: $\mathbf{y = f(x)}$ respectively $\mathbf{f : M_1 \to M_2}$

Forms of presentation:

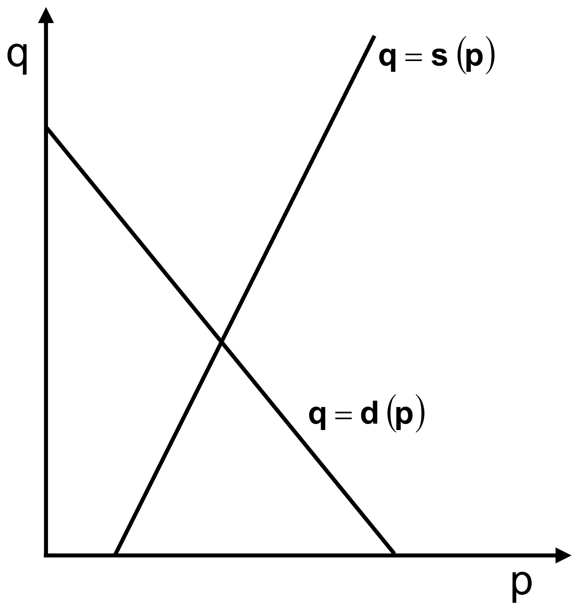
(a) value table

(b) graphical presentation (graph)

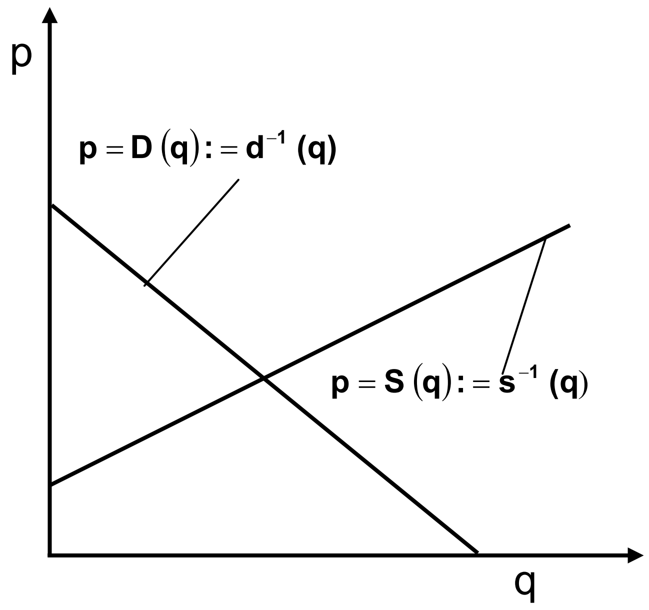
(c) functional equality $\mathbf{y = f(x)}$

Example 1: Supply- and demand function, production and consumption of a good (e.g. wheat) in a country; Quantity supplied and demanded depending on the price:

$$q = s(p)$$
$$q = d(p)$$

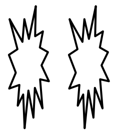


Mathematical presentation



Economical presentation

Example 2: growth of bacteria

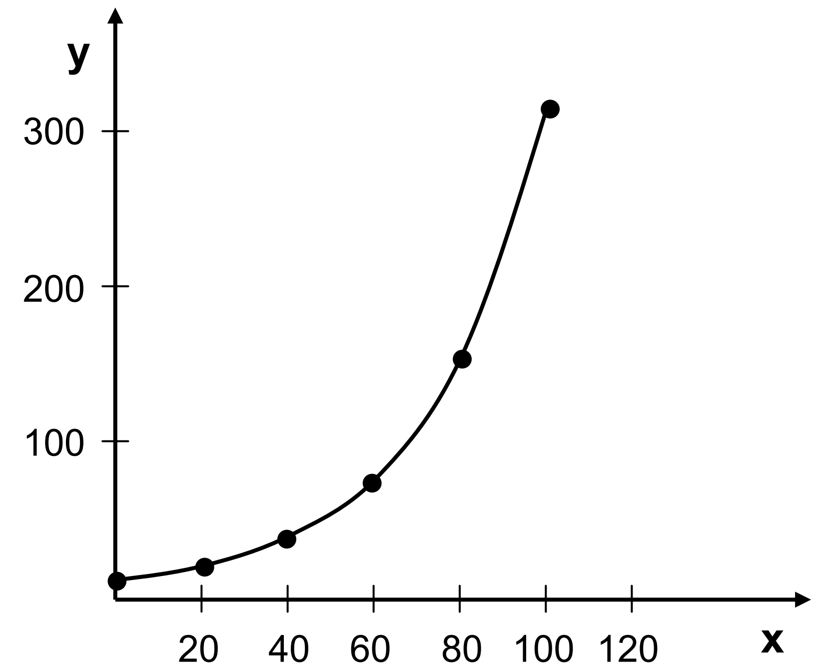
 → Division after 20 minutes → 

A nutritive substrate contains 10 bacteria in the beginning.
How many bacterias does the nutritive substrate contain 20, 40, 60 … minutes later?
How long does the development proceed in this regularity?

(a)

x	0	20	40	60	80	100	120	....
f (x)	10	20	40	80	160	320	640	

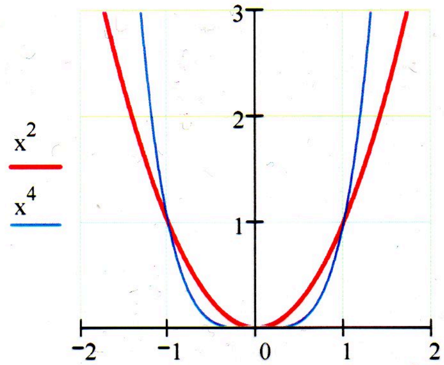
(b)

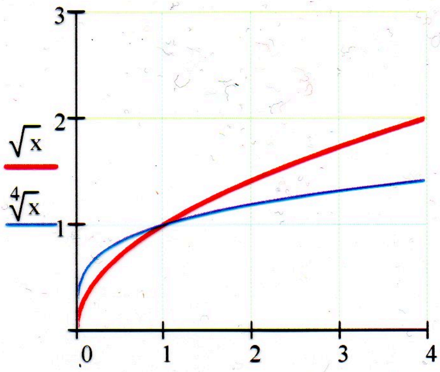


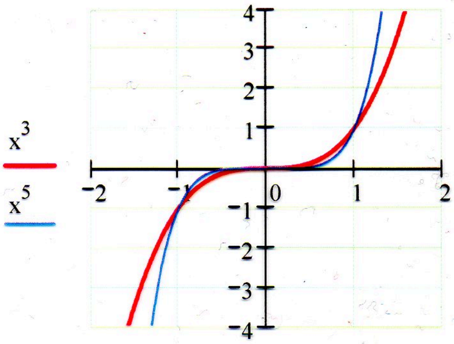
(c) $\mathbf{y = 10 \cdot 2^{\left(\frac{x}{20}\right)}}$

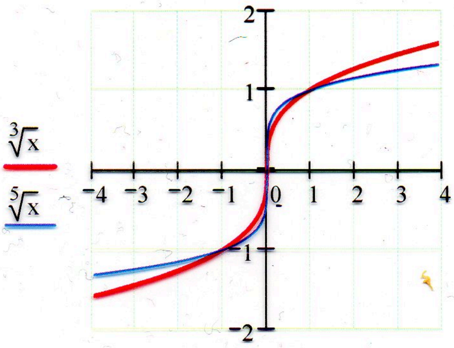
Graphs of basic functions y = f(x)

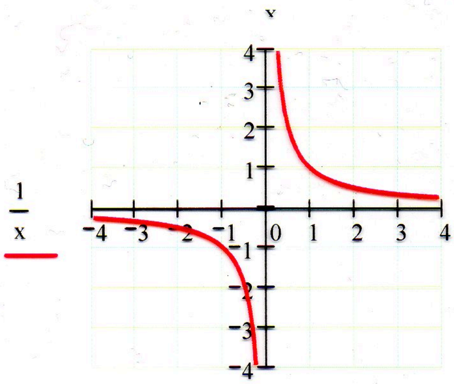
Darstellung der Grundfunktionen y = f(x)

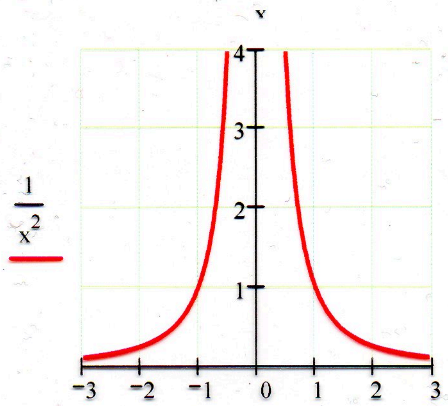


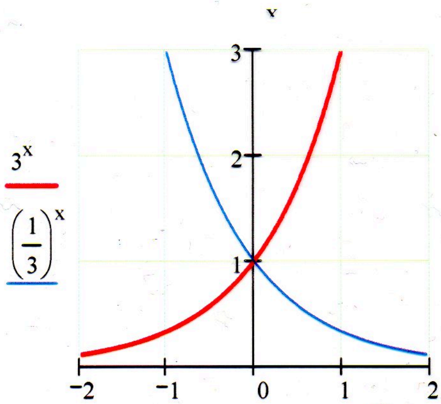


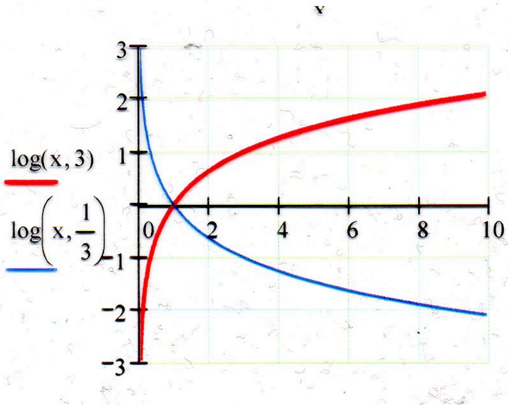








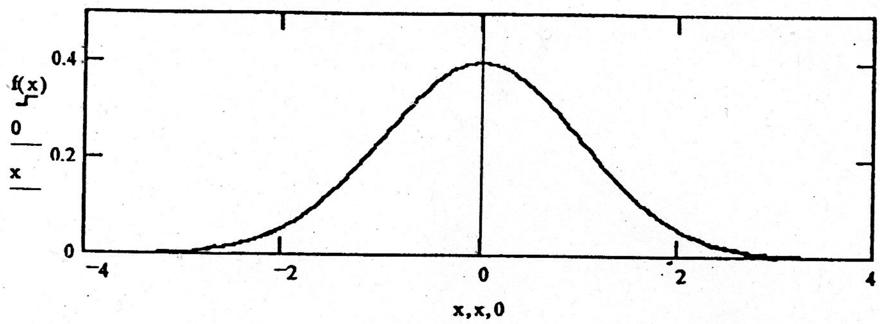




Gauss function

Gauß'sche Glockenkurve $f(x) := \frac{1}{\sqrt{2\cdot\pi}}\cdot e^{-\frac{x^2}{2}}$

$x := -4, -3.99 .. 4$

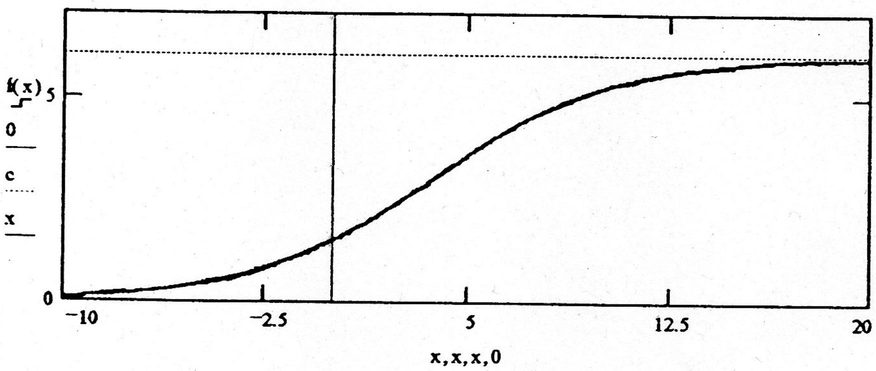


Logistic function

logistische Funktion $f(x) := \frac{c}{1 + a\cdot e^{-b\cdot x}}$

$a := 3 \quad b := 0.3 \quad c := 6$

$x := -10, -9.9 .. 20$

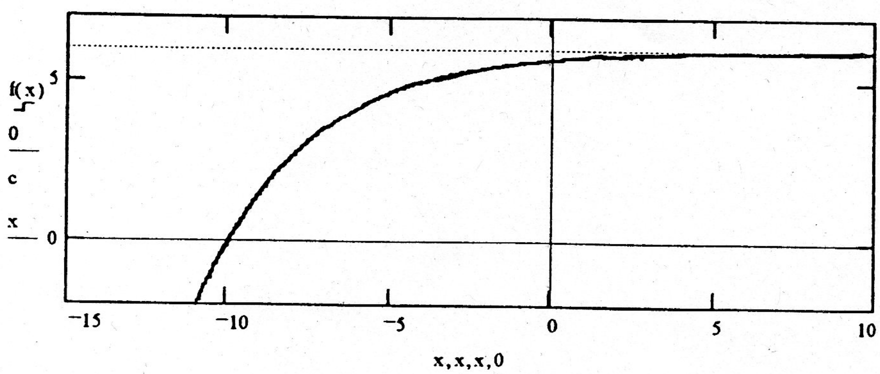


Mitscherlich function

Mitscherlich-Funktion $f(x) := c\cdot\left(1 - e^{-a - b\cdot x}\right)$

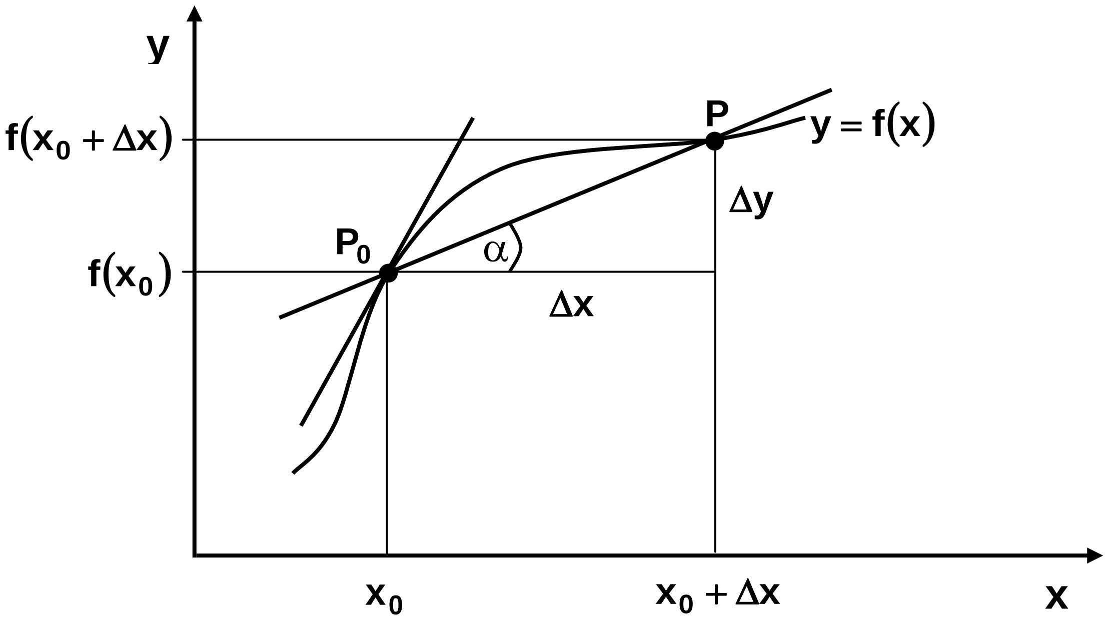
$a := 3 \quad b := 0.3 \quad c := 6$

$x := -20, -19.9 .. 10$



2.2 Differential calculus, integral calculus

Concerning many economical and other practical questions it will be of interest to know about the change behaviour of functions.



Difference quotient : $\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$

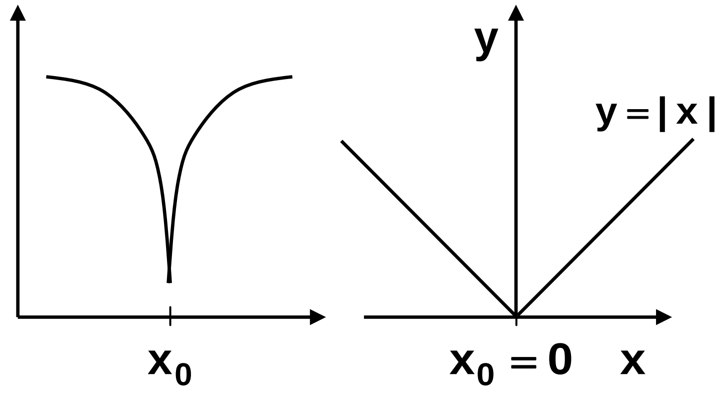
Transition to the limit value (from right hand and left hand)

Does the differential quotient

$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ exist?

no

f is not differentiable in x_0



yes

f is differentiable in x_0

The limit value is called derivative of the function f in point x_0

with the designation $f'(x_0)$

(gradient of the function in x_0;

gradient of the tangent to the function $f(x)$ in point x_0).

The function $\mathbf{y = f(x)}$ is called differentiable, if $\mathbf{f(x)}$ is differentiable in all points of the definition domain.

The derivatives in all points generate a function of the variable x, which is designated with $\mathbf{f'(x)}$ or shortly $\mathbf{f'}$ (following Lagrange), respectively with $\mathbf{\frac{dy}{dx}}$ (after Leibnitz).

Technology of differentiation:

Derivative of basic functions		
$f(x)$	$f'(x)$	
x^n	$n \cdot x^{n-1}$	$\begin{cases} \text{a)} & n \in N,\ x \in R \\ \text{b)} & n \in G,\ x \neq 0 \\ \text{c)} & n \in R,\ x > 0 \end{cases}$
e^x	e^x	
$\ln x$	$\frac{1}{x}$	
$\sin x$	$\cos x$	
$\cos x$	$-\sin x$	
a^x	$a^x \cdot \ln a$	
$\log_a x$	$\frac{1}{x \cdot \ln a}$	

+

Rule of derivation	
Sum rule	$[f(x) \pm g(x)]' = f'(x) \pm g'(x)$
Product rule	$[f(x) \cdot g(x)]' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
Quotient rule	$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x))}{(g(x))^2}$
Chain rule	$[g(f(x))]' = g'(f(x)) \cdot f'(x)$

⇓

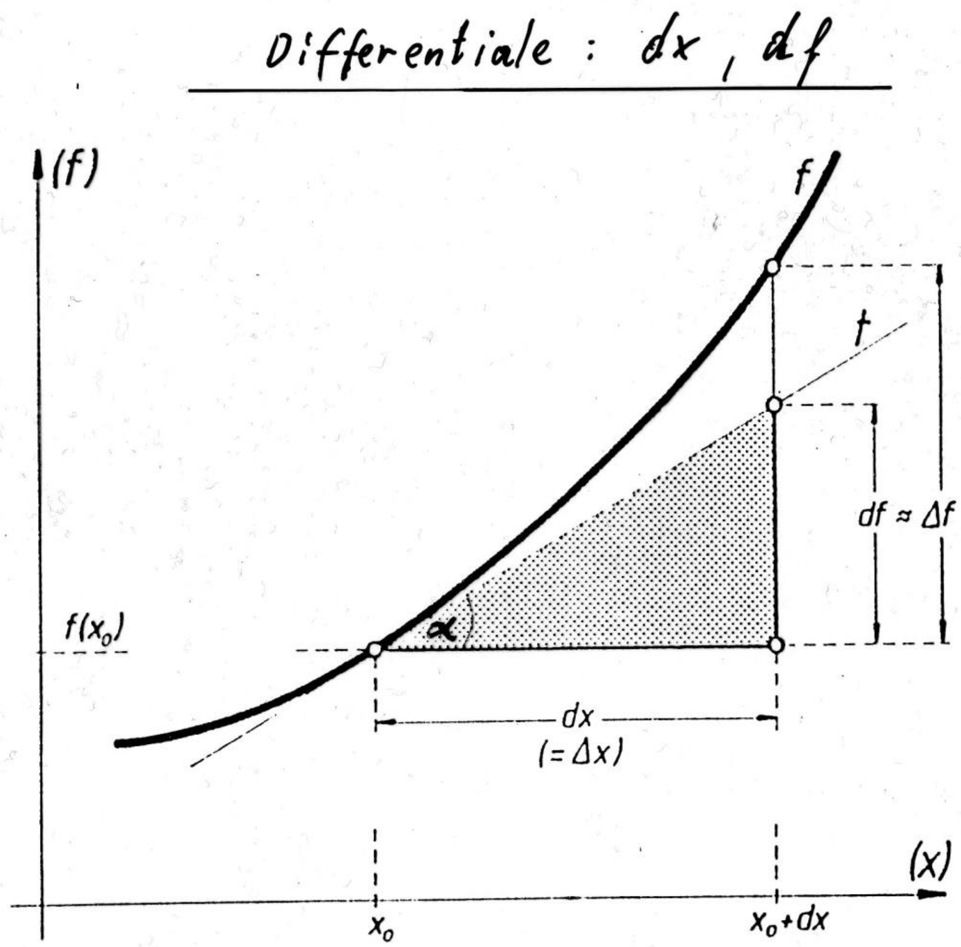
Derivation of „combined“ functions

Higher derivations:

Derivations of the first derivation **$f'(x) \Rightarrow f''(x)$** second derivation

Third derivation : **$f'''(x)$**

n^{th} derivation : **$f^{(n)}(x)$**



Application of derivative of functions:

a) Investigations of extreme values

local maximum/minimum at the point x_0

$(f'(x_0) = 0 \text{ and } f''(x_0) \neq 0)$

b) Curve sketching

$y = f(x)$ $\rightarrow$ Characteristics $\rightarrow$ graph of the function

c) Economics and natural sciences

Marginal function:

$P(x) = R(x) - C(x) \rightarrow \max$

Profit is defined as revenue minus costs

Necessary condition

$$P'(x) = R'(x) - C'(x) = 0$$

bzw. $R'(x) = C'(x)$

Marginal revenue is equal to marginal costs

Elasticity:

Price elasticity of demand (If the actual price changes by 1 %, by what percentage does the demand change?)

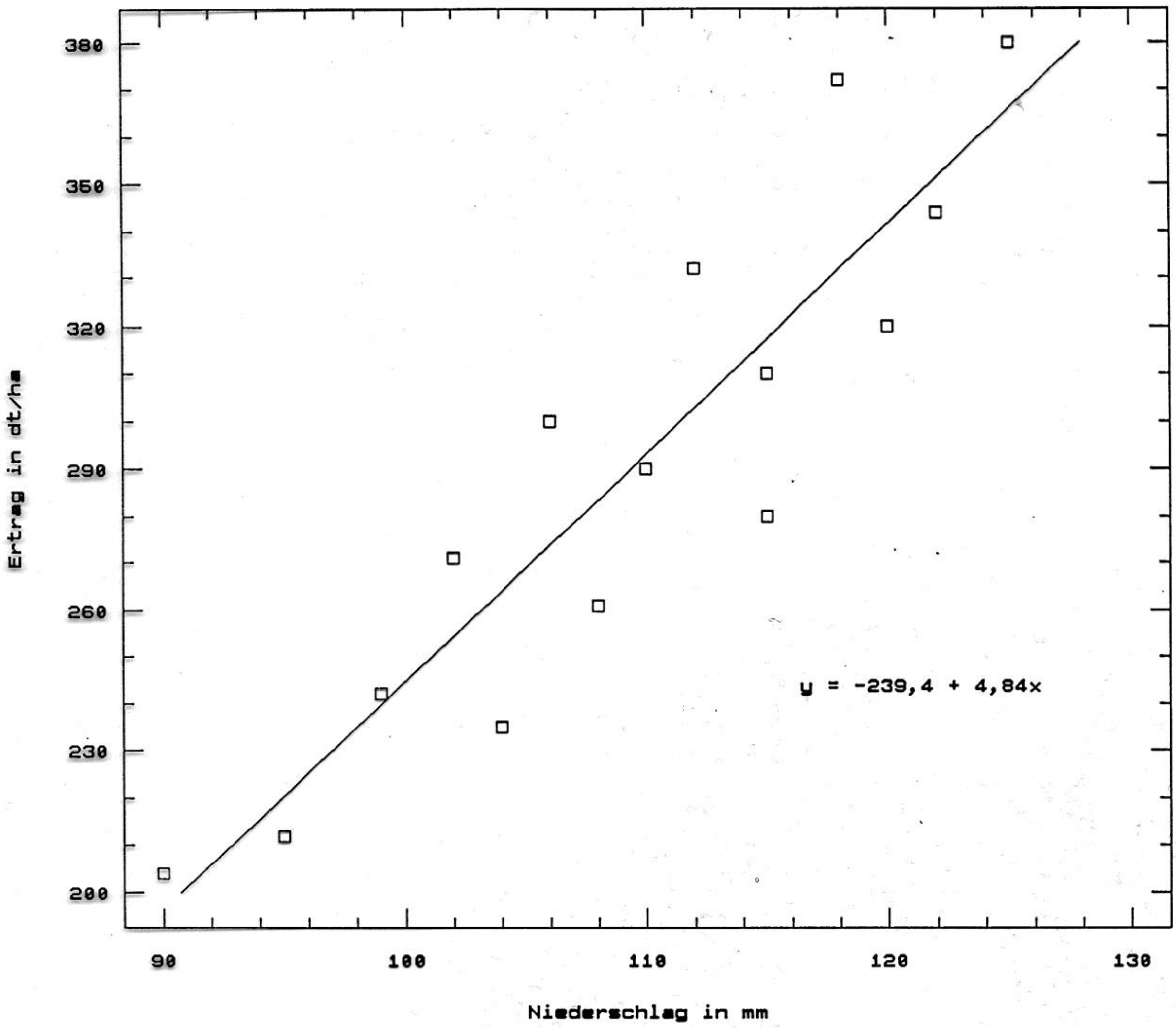
$$\varepsilon_{d,p} := d'(p) \cdot \frac{p}{d(p)}$$

Growth functions:

exponential; with saturation (logistical function,

sigmoid function (s-shape))

Analysis of trend and regression



Integral calculus

(1) Indefinite integral

Reverse of differentiation (more difficult)

$$\int f(x)\,dx = \{F(x) \mid F'(x) = f(x)\}$$

Set of all antiderivatives F of f (additive constant c)

(2) Definite integral (Riemann's Integral) defined as limit value

$$\int_a^b f(x)\,dx,$$

measures the area between the graph of f and the x-axis in the range of a to b.

(3) Calculation by using an antiderivative

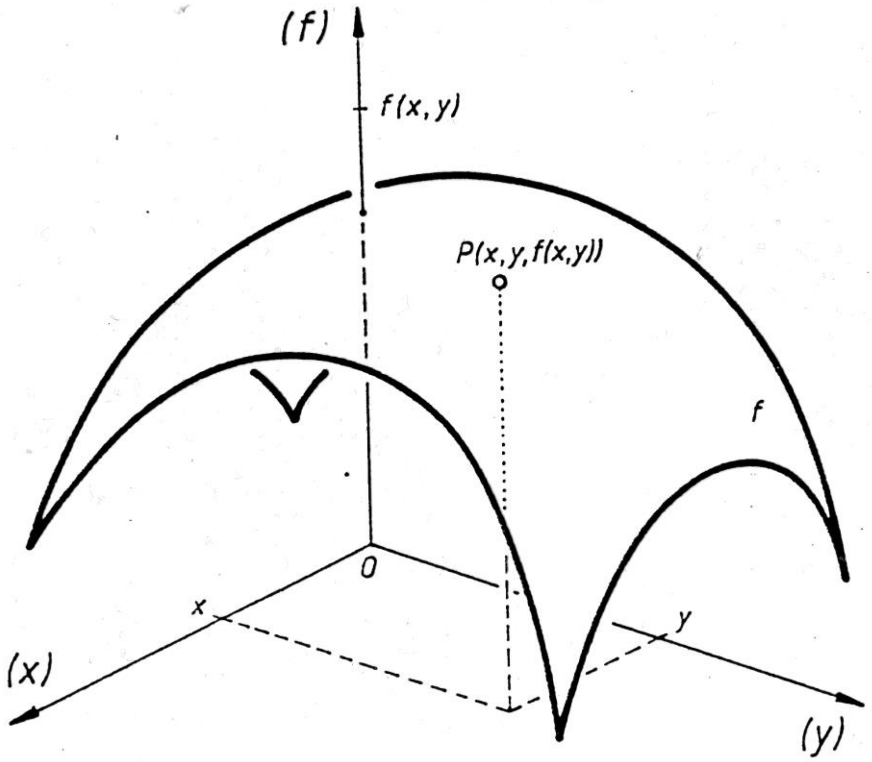
$$\int_a^b f(x)\,dx = F(b) - F(a) =: F(x)\Big|_a^b$$

Example: $\int_1^2 3x^2\,dx = x^3\Big|_1^2 = 2^3 - 1^3 = 8 - 1 = 7$

2.3 Multivariate functions

$$y = f(\underline{x}) = f(x_1 \dots , x_n) : Q \subset R^n \rightarrow R$$

Definite mapping of Q into R



2.3.1 Examples, Continuity

- Yield function
- Graph of the function is displayable as curved surface for n=2
- Paraboloid $y = x_1^2 + x_2^2$
- Objective function in the fodder-mix modell:
  minimal costs: $60x_1 + 45x_2 + 36x_3 = f(x_1, x_2, x_3)$
  is a linear function

Remarks concerning the continuity of functions

n = 1

f(x) continous in x_0: Is the distance of **x and x_o** small, the distance of **f(x) and f(x_0)** is small as well.

$$\Updownarrow$$

$$\lim_{x \to x_0} f(x) = f(x_0)$$

n > 1 $\underline{x} = (x_1, x_2, ..., x_n) \in R^n$ $\underline{x}^0 = (x_1^0,\ x_2^0, ..., x_n^0) \in R^n$

f($\underline{x}$) continous in $\underline{x}^0$: Is the distance of **$\underline{x}$ and $\underline{x}^0$** small, the distance of **f($\underline{x}$) and f($\underline{x}^0$)** is small as well.

$$\Updownarrow$$

$$\lim_{\underline{x} \to \underline{x}^0} f(\underline{x}) = f(\underline{x}^0)$$

Concerning this, the distance of n-tuples is defined using the Euclidean norm.

$$\| \underline{x} - \underline{x}^0 \| := \sqrt{\sum_{i=1}^{n} (x_i - x_i^0)^2}\ .$$

Heuristics: n = 1 Drawing of the curve without any break

n = 2 „Curved surface without any hole or crack".

2.3.2 Partial derivative

Gradient of the function in direction of the axes

Partial derivative:

Derivative of the function $\mathbf{f(\underline{x}) = f(x_1, x_2, ..., x_n)}$ with respect **to one of those variables in each case**, whereby the other variables are concerned as constants.

Also the definition of the partial derivative is done at first at the local point $\underline{\mathbf{x}}^{\mathbf{0}} \in \mathbf{R}^{\mathbf{n}}$ by using a limit:

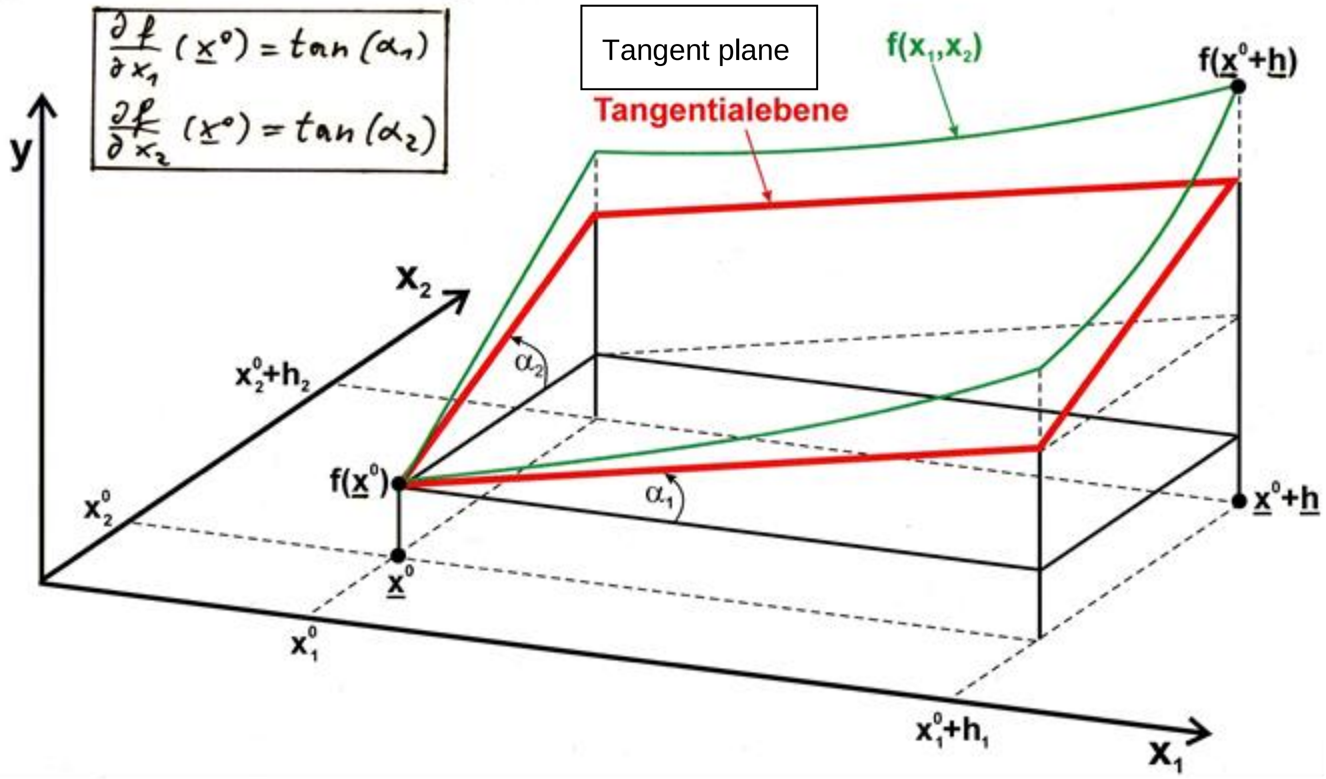
$$\lim_{h \to 0} \frac{f(x_1^0, ..., x_i^0 + h, ..., x_n^0) - f(x_1^0, ..., x_i^0, ..., x_n^0)}{h} \quad \text{for } i = 1, ..., n.$$

Designation: $\frac{\partial f}{\partial x_i}$ (also $\frac{\partial}{\partial x_i} f$, $\frac{\partial y}{\partial x_i}$ **respectively** f_{x_i}, f_i)

Geometrical interpretation:

Gradient of the function in direction of the axes.

Geometrische Interpretation der partiellen Ableitungen



2. Partial derivatives (Second order partial derivatives):

$$\frac{\partial^2 f}{\partial x_i \, \partial x_j} := \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)$$ as abbreviated form also f_{ij};

This means that the partial derivative of the function with respect to x_i, is partially derivated again with respect to the variable x_j.

Altogether there are n^2 second order derivative functions:

$$\begin{matrix} \frac{\partial^2 f}{\partial x_1^2}, & \frac{\partial^2 f}{\partial x_1 \, \partial x_2}, & \ldots, & \frac{\partial^2 f}{\partial x_1 \, \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \, \partial x_1}, & \frac{\partial^2 f}{\partial x_2^2}, & \ldots, & \frac{\partial^2 f}{\partial x_2 \, \partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n \, \partial x_1}, & \frac{\partial^2 f}{\partial x_n \, \partial x_2}, & \ldots, & \frac{\partial^2 f}{\partial x_n^2} \end{matrix}$$

Example 1: $y = f(x_1, x_2) = x_1^2 + x_2^2$

$$\frac{\partial f}{\partial x_1} = 2x_1 \quad , \quad \frac{\partial f}{\partial x_2} = 2x_2$$

Example 2: $f(x_1, x_2) = 5 + 2x_1 + 5x_1^2 + 8x_1 x_2 + 7x_2 + 5x_2^2$

$$\frac{\partial f}{\partial x_1} = 2 + 10x_1 + 8x_2 \quad , \quad \frac{\partial f}{\partial x_2} = 8x_1 + 7 + 10x_2 ,$$

$$\frac{\partial^2 f}{\partial x_1 \, \partial x_1} = 10 \qquad \frac{\partial^2 f}{\partial x_1 \, \partial x_2} = 8$$

$$\frac{\partial^2 f}{\partial x_2 \, \partial x_1} = 8 \qquad \frac{\partial^2 f}{\partial x_2 \, \partial x_2} = 10$$

2.3.3 Local extrema of multivariate functions

(1) One variable: $\mathbf{x} \in \mathbf{R}, \quad \mathbf{y} = \mathbf{f(x)}$,

we are looking for a local maximum respectively minimum of the function f(x)

necessary condition: $\mathbf{f'(x) = 0}$,

sufficient condition: $\mathbf{f''(x) < 0}$ local maximum

bzw. $\mathbf{f''(x) > 0}$ local minimum.

(2) n variables: $\mathbf{(x_1, x_2, ..., x_n) = \underline{x} \in R^n}$

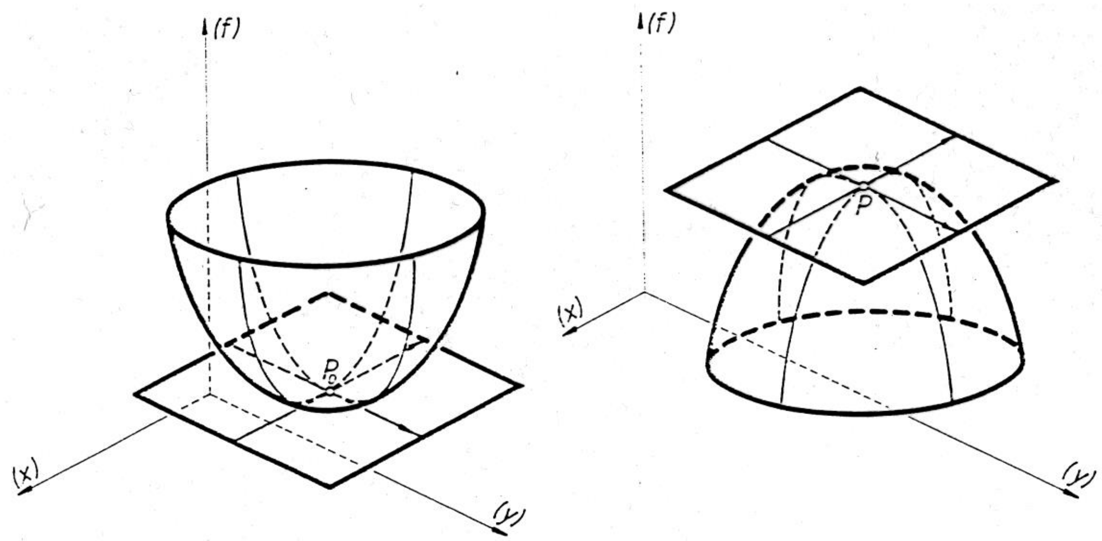
Definition:

> The function f(x) has a local maximum(respectively minimum) at an inner point $\underline{\mathbf{x}}^{\mathbf{0}}$ of the definition range if a real number $\delta > \mathbf{0}$ exists with $\mathbf{f(\underline{x}) \le f(\underline{x}^0)}$ $\mathbf{(respectively\ f(\underline{x}) \ge f(\underline{x}^0))}$ for every $\underline{\mathbf{x}} \in \mathbf{R^n}$ with $\| \underline{\mathbf{x}} - \underline{\mathbf{x}}^{\mathbf{0}} \| < \delta$.

Collective term: local extremum

$\left\{ \underline{\mathbf{x}} \in \mathbf{R^n} \mid \| \underline{\mathbf{x}} - \underline{\mathbf{x}}^{\mathbf{0}} \| < \delta \right\}$ is called an open sphere around $\underline{\mathbf{x}}^{\mathbf{0}} \in \mathbf{R^n}$

Local maximum/ Local Minimum



Necessary condition for relative extrema:

Theorem: If $\mathbf{f(\underline{x})}$ has a local extremum at point $\underline{\mathbf{x}}^{\mathbf{0}} \in \mathbf{R}^{\mathbf{n}}$ and if all partial derivatives $\frac{\partial \mathbf{f}}{\partial \mathbf{x_i}}$ exist at this point, than $\frac{\partial \mathbf{f}}{\partial \mathbf{x_i}}(\underline{\mathbf{x}}^{\mathbf{0}}) = \mathbf{0} \quad \forall \mathbf{i} = \mathbf{1, ..., n}$.

We consider n equations. The solutions of this system of equations are called stationary points.

Furthermore let n = 2, that means we have two variables $\mathbf{x_1, x_2}$.

So there is a geometrical interpretation of the necessary conditions:

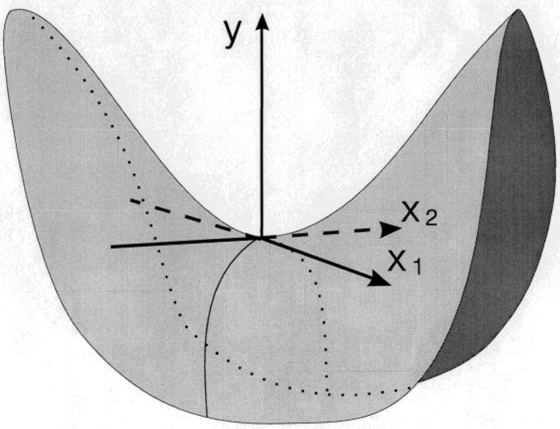
the tangent plane of $\mathbf{f(x_1, x_2)}$ at point $\underline{\mathbf{x}}^{\mathbf{0}} = (\mathbf{x_1^0}, \mathbf{x_2^0})$ is parallel to the $\mathbf{x_1, x_2}$ −plain.

In this case, we also need a sufficient condition for the local extremum.

Example: Saddle surface $\mathbf{f(x_1, x_2) = x_1 \cdot x_2}$

Sattelfläche $f(x_1, x_2) = x_1 \cdot x_2$

Saddle surface



Sufficient condition for a relative extremum:

Let $f(x_1,x_2)$ be defined and continuous in a neighbourhood of a stationary point $\underline{x}^0 = (x_1^0, x_2^0)$ and let us suppose that all first and second partial derivatives exist and are continuous as well.

Theorem: $f(x_1,x_2)$ has a local minimum at the stationary point

$\underline{x}^0 = (x_1^0, x_2^0)$ if

(1) $$D = \frac{\partial^2 f}{\partial x_1\,\partial x_1}(\underline{x}^0)\cdot\frac{\partial^2 f}{\partial x_2\,\partial x_2}(\underline{x}^0) - \left[\frac{\partial^2 f}{\partial x_1\,\partial x_2}(\underline{x}^0)\right]^2 > 0$$

and

(2) $$\frac{\partial^2 f}{\partial x_1\,\partial x_1}(\underline{x}^0) > 0.$$

If D > 0 and $\frac{\partial^2 f}{\partial x_1\,\partial x_1}(\underline{x}^0) < 0$, then f has a local maximum at $\underline{x}^0$.

In Example 2 from section 2.3.2 we defined all first and second partial derivatives for the function

$$f(x_1, x_2) = 5 + 2x_1 + 5x_1^2 + 8x_1 x_2 + 7x_2 + 5x_2^2 \ .$$

The necessary condition is the linear equality system

$$2 + 10x_1 + 8x_2 = 0$$

$$7 + 8x_1 + 10x_2 = 0 \ .$$

The solution is the stationary point $\underline{x}^0 = (1, -\frac{3}{2})$.

The sufficient condition is

(1) $D = 10 \cdot 10 - 8^2 = 36 > 0$ and

(2) $\frac{\partial^2 f}{\partial x_1 \, \partial x_1} = 10 > 0$.

(1) makes sure there is a local extremum at point $\underline{x}^0$,

(2) shows that the local extremum is a local minimum.

2.4 Generalizations, applications

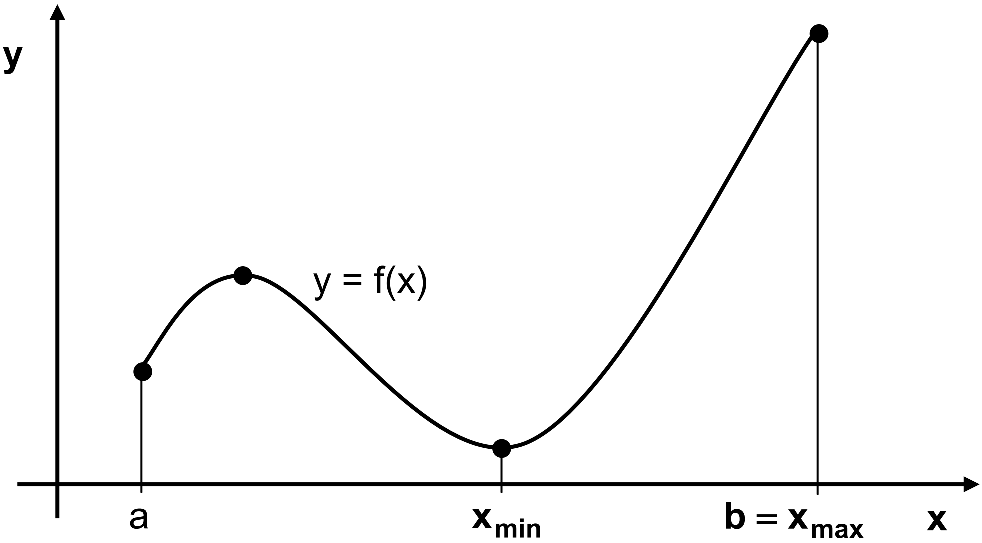
a) Absolute extremum

Definition:

A function $\mathbf{f(\underline{x}) : M \subset R^n \to R}$ has an absolute maximum at point $\mathbf{\underline{x}^0} \in \mathbf{M}$, if $\mathbf{f(\underline{x}^0) \geq f(\underline{x})} \quad \forall\ \mathbf{\underline{x} \in M}$.

n = 1

M = [a.b]



In general the theorem of Weierstrass is valid:

If $\mathbf{f(\underline{x})}$ is a continous function in a restricted and closed set M, then an absolute maximum respectively minimum of f regarding M exists.

Methodology:

- consider $\mathbf{f(\underline{x})}$ in a compact (restricted and closed) set, e.g. a n-dimensionally cuboid,
- determine all local extremes,
- compare those to the values of f at the margin of set M.

b) Taylor's theorem

Describing the value of a function in the surrounding of a known point by using the (partial) derivative.

$\mathbf{n = 1: x_0 \in R, f(x_0)}$ and the derivatives $\mathbf{f^{(k)}(x_0)}$

until the order m are given, considering

$$\mathbf{x_0 + h \in R,}$$

$$\mathbf{f(x_0 + h) = \sum_{k=0}^{m} \frac{f^{(k)}(x_0)}{k\,!} \cdot h^k + R_{m+1}}$$

$$\mathbf{n > 1: \; \underline{x}^0 = (x_1^0, x_2^0, \; ..., x_n^0) \in R^n,}$$

$$\mathbf{f(\underline{x}) : R^n \to R, \qquad \underline{h} = (h_1, h_2, ..., h_n) \in R^n}$$

The first and second order „partial" derivatives of f in point $\mathbf{\underline{x}^0}$ are given,

$$\mathbf{f(\underline{x}^0 + \underline{h}) = f(x_1^0 + h_1, x_2^0 + h_2, ..., x_n^0 + h_n)}$$

$$\mathbf{= f(\underline{x}^0) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\underline{x}^0) \cdot h_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \, \partial x_j} (\underline{x}^0) \cdot h_i \cdot h_j + R}$$

c) Relative constrained extremum

$\mathbf{f(\underline{x}): R^n \to R, \quad g_i(\underline{x}): R^n \to R, i = 1, ..., m}$,

let the partial derivatives be continuous,

$$\mathbf{f(\underline{x}) \to min\ (max)}$$

Under the conditions:

$$\begin{aligned} g_1(\underline{x}) &= 0 \\ g_2(\underline{x}) &= 0 \\ &\vdots \\ g_m(\underline{x}) &= 0 \end{aligned}$$

We consider for the **Lagrange-Funktion**

$$L(\underline{x}, \underline{\lambda}) := f(\underline{x}) + \lambda_1 g_1(\underline{x}) + ... + \lambda_m g_m(\underline{x})$$

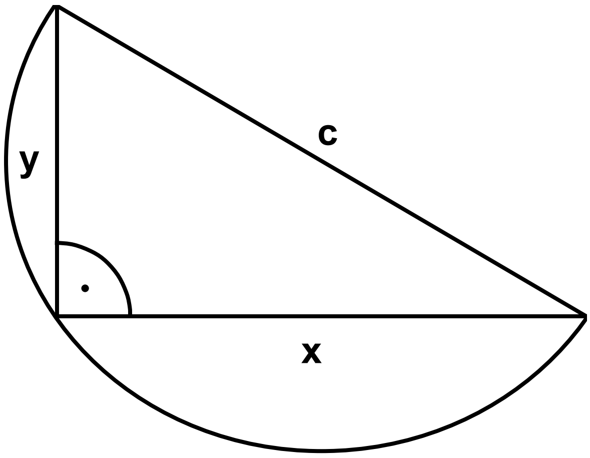
the necessary conditions for a relative extremum:

$$\frac{\partial L}{\partial x_i} = 0 \quad , \text{ für } i = 1, ..., n,$$

$$\frac{\partial L}{\partial \lambda_j} = 0 \quad , \text{ für } j = 1, ..., m$$

and solve this system of equations.

Example:



Given the hypotenuse c, for which x,y the surface of the triangle is at its maximum?

$$f(x,y) = \frac{x \cdot y}{2} \rightarrow \max$$

Under the conditions $x^2 + y^2 = c^2$

$$L(x,\ y,\ \lambda) = \frac{x \cdot y}{2} + \lambda \cdot (x^2 + y^2 - c^2)$$

$$\frac{\partial L}{\partial x} = \frac{y}{2} + 2\ \lambda\, x = 0 \qquad (1)$$

$$\frac{\partial L}{\partial y} = \frac{x}{2} + 2\ \lambda\, y = 0 \qquad (2)$$

(1) $\rightarrow \quad \lambda = -\frac{y}{4x}$

add in (2): $\quad \frac{x}{2} - \frac{y^2}{2x} = 0 \quad \rightarrow \quad x^2 = y^2$

with $\quad x^2 + y^2 = c^2 \quad (\frac{\partial L}{\partial \lambda} = 0)$ follows $x^2 = y^2 = \frac{c^2}{2}$,

this means x = y (isosceles triangle).

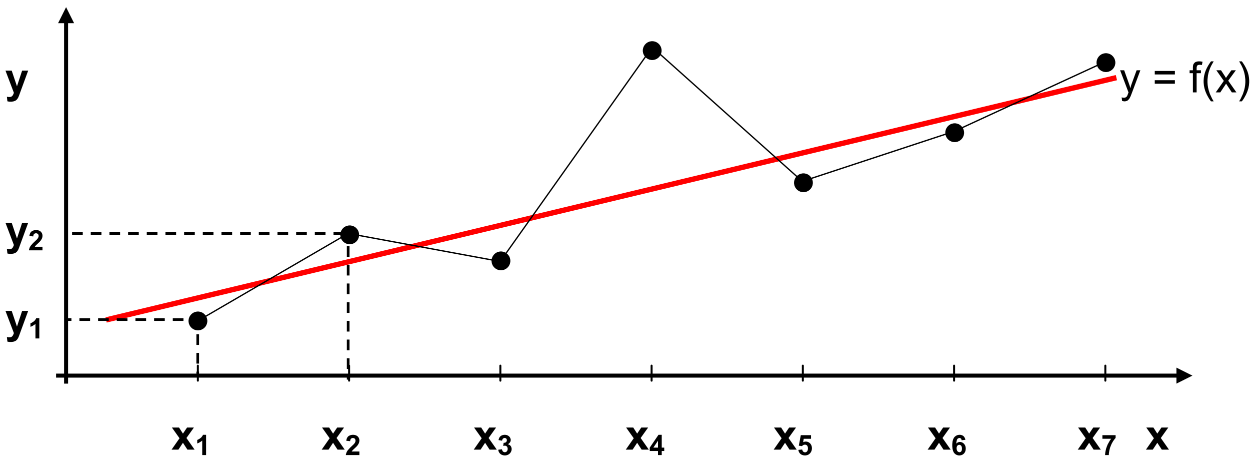
d) Analysis of trend and regression

- Investigation of behaviour respectively of change of certain data or value– economical, biological i.a.

  e.g. Gross national product, saving deposit, annual milk consumption of the population etc.

- List of time series**:**

year x	1994	1995	1996	1997	1998	1999	2000
data y	y_1	y_2	y_3	y_4	y_5	y_6	y_7



Requested: functional „dependency“ of y regarding x

- Different functional models are possible:
  - assumption of a linear connection; we are looking for a linear function $y = ax + b$, so that the given points „are in the surrounding of the function” this means we are looking for the parameters a and b; such a function is called linear trend function (it is also called adjustment of data, fitting respectively equalization calculus)

-Assumption: the function we are looking for is a polynomial

$$\mathbf{y} = \sum_{\mathbf{i=0}}^{\mathbf{n}} \mathbf{a_i\, x^i},$$

this means we are looking for n, a_0, a_1, ..., a_n;

n = 2 quadratical adjustment

n = 3 cubical adjustment.

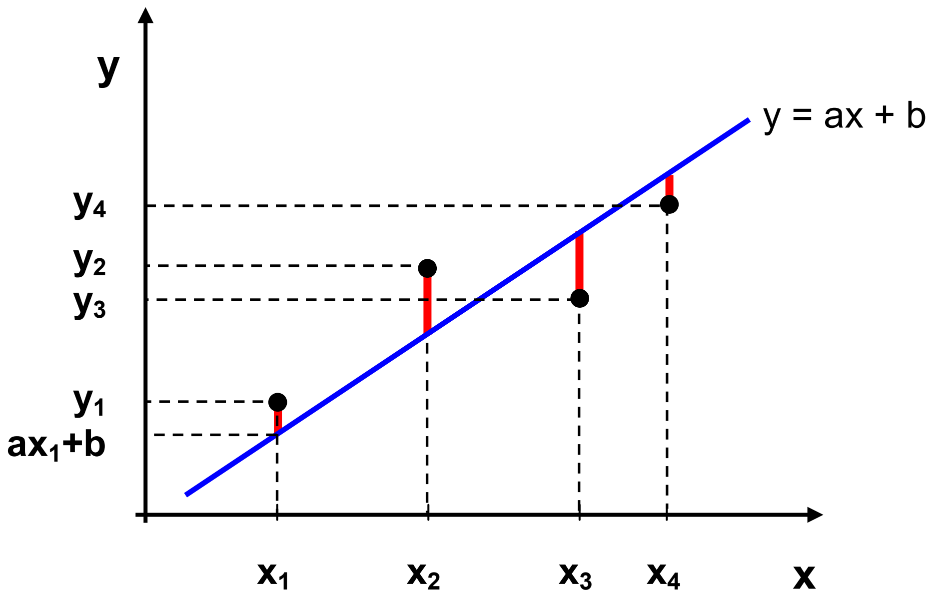
- we are looking for a good adjustment using an exponential function (e.g. regarding heavy increase) $y = a\cdot e^{bx}$, this means we are looking for the corresponding parameters a and b.

- So far we considered time steps x_1, x_2, x_3, ..., with similar gaps (e.g. annual measures).⇒ **Trend**
  Now we consider two features (random parameters) x and y.
  ⇒ **Regression**

e.g. (1) x – weight of an animal in a population
y – size of the animal.
We are talking about a random sample x_i, y_i for i = 1, 2,...,k

(2) x – precipitation amount
y – harvest of an agricultural good(ZR).

Question: Does a connection exist between x and y?
Of which type is this connection, maybe linear?



$$\varepsilon_1 = y_1 - (ax_1 + b) \rightarrow \varepsilon_1^2 = (y_1 - ax_1 - b)^2$$
$$\varepsilon_2 = y_2 - (ax_2 + b) \rightarrow \varepsilon_2^2 = (y_2 - ax_2 - b)^2$$
$$\varepsilon_3 = ax_3 + b - y_3 \rightarrow \varepsilon_3^2 = (y_3 - ax_3 - b)^2$$
$$\varepsilon_4 = ax_4 + b - y_4 \rightarrow \varepsilon_4^2 = (y_4 - ax_4 - b)^2$$

$$\text{Sum:} \quad \sum_{i=1}^{n} (y_i - ax_i - b)^2$$

We do now have a function with two variables (a is the ascent and b is the intersection point with the y-axis of the requested function y = ax + b), which shall be minimized:

$$F = F(a,b) = \sum_{i=1}^{n} (y_1 - ax_i - b)^2 \rightarrow \min.$$

The method of least squares

$$F(a,b) = \sum_{i=1}^{n}(y_i - a \cdot x_i - b)^2 \rightarrow \text{Min}$$

at a given random sample x_i , y_i for i = 1,...,n

$$\frac{\partial F}{\partial b} = \sum_{i=1}^{n} 2(y_i - ax_i - b)\cdot(-1) = -2\sum_{i=1}^{n}(y_i - ax_i - b)$$

$$-2\sum_{i=1}^{n}(y_i - ax_i - b) = 0 \qquad \Big| :(-2)$$

$$\sum_{i=1}^{n}(y_i - ax_i - b) = 0$$

$$\sum_{i=1}^{n} y_i - \sum_{i=1}^{n} ax_i - \sum_{i=1}^{n} b = 0 \qquad \Big| \sum_{i=1}^{n} b = n \cdot b$$

$$\sum_{i=1}^{n} y_i - a\sum_{i=1}^{n} x_i = nb \qquad \Big| :n$$

$$b = \overline{y} - a\overline{x} \qquad \Big| \overline{x} = \tfrac{1}{n}\sum_{i=1}^{n} x_i \qquad \overline{y} = \tfrac{1}{n}\sum_{i=1}^{n} y_i \qquad \text{average values}$$

The function y=ax+b intersects the point

$$(\overline{x}, \overline{y})$$

$$F(a,b)=\sum_{i=1}^{n}(y_i-a\cdot x_i-b)^2 \rightarrow \text{Min}$$

$$\frac{\partial F}{\partial a}=\sum_{i=1}^{n}2(y_i-ax_i-b)\cdot(-x_i)=-2\sum_{i=1}^{n}(y_i-ax_i-b)x_i$$

$$-2\sum_{i=1}^{n}(y_i-ax_i-b)x_i=0 \qquad \Big|:(-2)$$

$$\sum_{i=1}^{n}(y_i-ax_i-b)x_i=0$$

$$\sum_{i=1}^{n}y_ix_i-\sum_{i=1}^{n}ax_i^2-\sum_{i=1}^{n}bx_i=\sum_{i=1}^{n}y_ix_i-a\sum_{i=1}^{n}x_i^2-b\sum_{i=1}^{n}x_i=0$$

$$\sum_{i=1}^{n}y_ix_i-a\sum_{i=1}^{n}x_i^2-(\overline{y}-a\overline{x})\sum_{i=1}^{n}x_i=0$$

$$\sum_{i=1}^{n}y_ix_i-\overline{y}\sum_{i=1}^{n}x_i=a\Big(\sum_{i=1}^{n}x_i^2-\overline{x}\sum_{i=1}^{n}x_i\Big) \qquad \Big|:\Big(\sum_{i=1}^{n}x_i^2-\overline{x}\sum_{i=1}^{n}x_i\Big)$$

$$a=\frac{\sum_{i=1}^{n}y_ix_i-\overline{y}\sum_{i=1}^{n}x_i}{\Big(\sum_{i=1}^{n}x_i^2-\overline{x}\sum_{i=1}^{n}x_i\Big)} \qquad\qquad b=\overline{y}-\frac{\sum_{i=1}^{n}y_ix_i-\overline{y}\sum_{i=1}^{n}x_i}{\Big(\sum_{i=1}^{n}x_i^2-\overline{x}\sum_{i=1}^{n}x_i\Big)}\cdot\overline{x}$$

$$\left(a=\frac{\sum_{i=1}^{n}(x_i-\overline{x})(y_i-\overline{y})}{\sum_{i=1}^{n}(x_i-\overline{x})^2} \qquad , \qquad y=\overline{y}+a(x-\overline{x})\right)$$

Checking the sufficient condition

$$\frac{\partial^2 F}{\partial a \partial a} = 2\sum_{i=1}^{n} x_i^{\,2} > 0 \qquad \frac{\partial^2 F}{\partial a \partial b} = \frac{\partial^2 F}{\partial b \partial a} = 2\sum_{i=1}^{n} x_i \qquad \frac{\partial^2 F}{\partial b \partial b} = 2n$$

$$D = 2\sum_{i=1}^{n} x_i^{\,2} \cdot 2n - 4(\sum_{i=1}^{n} x_i)^2 = 4n \cdot \left(\sum_{i=1}^{n} x_i^{\,2} - \tfrac{1}{n}(\sum_{i=1}^{n} x_i)(\sum_{i=1}^{n} x_i) \right)$$

$$= 4n \cdot \left(\sum_{i=1}^{n} x_i^{\,2} - \overline{x} \cdot \sum_{i=1}^{n} x_i \right)$$

$$= 4n \cdot \left(\sum_{i=1}^{n} x_i^{\,2} - n \cdot \overline{x}^2 \right) \qquad \Big| \text{ see (*)}$$

$$= 4n \cdot \left(\sum_{i=1}^{n} (x_i - \overline{x})^2 \right) > 0$$

(*) $$\sum_{i=1}^{n} (x_i - \overline{x})^2 = \sum_{i=1}^{n} (x_i^{\,2} - 2x_i\overline{x} + \overline{x}^2) = \sum_{i=1}^{n} x_i^{\,2} - 2\overline{x}\sum_{i=1}^{n} x_i + n\overline{x}^2$$

$$= \sum_{i=1}^{n} x_i^{\,2} - 2\overline{x} \cdot n\overline{x} + n\overline{x}^2 = \sum_{i=1}^{n} x_i^{\,2} - 2n\overline{x}^2 + n\overline{x}^2 = \sum_{i=1}^{n} x_i^{\,2} - n\overline{x}^2$$

3. Elements of linear algebra

3.1 Matrices and vectors

Example: animal stocks (in a region):

3 farms: milking cows, pig- and cattle fattening

Farm №.	M	P	C
1	**34**	**2**	**14**
2	**120**	**-**	**-**
3	**150**	**40**	**30**

Rectangular scheme of numbers MATRIX

$$\Rightarrow \begin{pmatrix} 34 & 2 & 14 \\ 120 & 0 & 0 \\ 150 & 40 & 30 \end{pmatrix}$$

3.1.1 Definition of the term matrix

Definition: A (m, n)-matrix is a system of $\mathbf{m \cdot n}$ numbers $\mathbf{a_{ik}}$ (i = 1, 2,..., m; k = 1, 2, ..., n), which are organized in a rectangular scheme of m lines and n columns.

$$\begin{pmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & & & a_{2n} \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ a_{m1} & a_{m2} & \cdot & \cdot & a_{mn} \end{pmatrix}$$

The numbers in the scheme are called elements of the matrix.

Symbolism: $\underline{\mathbf{A}} = (a_{ij})_{i=1 \dots m \quad j=1, \dots n} = (a_{ij})_{(m,n)}$

i – row index

j – column index

Where is a_{32}?

Type ($\underline{A}$): = (m, n),   m - number of rows
n - number of columns

Abbreviated, it is also possible to say $\mathbf{m \times n}$-matrix.

3.1.2 Special matrices

1) Matrix with only one column (Typ: (m, 1)): **column vector**,
   Matrix with only one row (Typ: (1, n)): **row vector**.
   Concerning vectors: elements → coordinates
   Number of elements → dimension of the vector

$$\underline{a} = \begin{pmatrix} \mathbf{a_1} \\ . \\ . \\ \mathbf{a_m} \end{pmatrix}$$ m is the dimension of a.

   Row vector $\underline{b}$ = $(b_1 \dots b_n)$,
   (Also: $\vec{\mathbf{a}}$)

   Example!

2) **Quadratical matrix**: Matrix of the type: (n, n)

3) **Diagonal matrix**: $a_{ij} = 0$ for $i \neq j$.

$$\begin{pmatrix} \mathbf{a_{11}} & \mathbf{0} & . & . & . & \mathbf{0} \\ \mathbf{0} & \mathbf{a_{22}} & \mathbf{0} & & & . \\ . & \mathbf{0} & . & \mathbf{0} & & . \\ . & & \mathbf{0} & . & \mathbf{0} & . \\ . & & & \mathbf{0} & . & \mathbf{0} \\ \mathbf{0} & . & . & . & \mathbf{0} & \mathbf{a_{nn}} \end{pmatrix}$$

   Main diagonal

Example!

4) $\underline{E} = (e_{ij})_{n,n}$: $e_{ij} = 0$ for i, j : $i \neq j$

$e_{ii} = 1$ for i = 1, ..., n

$\underline{E}$ is called n'th **identity matrix** (of n^{th} order); in literature also $\underline{\mathbf{I}}$

The columns of the identity matrix $\underline{E}$ are called unit vectors.

$$\underline{\mathbf{e}}_1 = \begin{pmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}, \underline{\mathbf{e}}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}, \cdots, \underline{\mathbf{e}}_i = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}, \cdots, \underline{\mathbf{e}}_n = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{pmatrix}$$

I^{th} coordinate

5) Upper triangular matrix

$$\begin{pmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1n} \\ 0 & a_{22} & & & & \\ \cdot & \cdot & a_{33} & & & \cdot \\ \cdot & & 0 & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & 0 & a_{nn} \end{pmatrix}$$

3.1.3 Relations between matrices respectively vectors

$\underline{A} = \underline{B}$: (1) the same type.

(2) the corresponding elements are equal.

Example:

$$\begin{pmatrix} 4 & 2 & 100 \\ 10 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 100 \\ 10 & 0 & -2 \end{pmatrix};$$

$$\underline{u} = \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix} \quad \Leftrightarrow \quad \begin{matrix} u_1 & = & -2 \\ u_2 & = & 1 \\ u_3 & = & 5 \end{matrix}$$

$\underline{A} \leq \underline{B}$: (1) the same type.

(2) $a_{ij} \leq b_{ij}$

analogue $\underline{A} < \underline{B}$

Example!

3.1.4 Transpose of a matrix

$\underline{A}$ → write the rows as columns → $\underline{A}^T$ (also $\underline{A}'$)

Example!

If $\underline{A}^T = \underline{A}$, then A is a symmetric matrix ($a_{ij} = a_{ji}$).

Example!

3.1.5 Operations of matrices

Addition:

$\underline{A}$, $\underline{B}$ with type: (m, n),

A matrix $\underline{C}$ with $c_{ij} = a_{ij} + b_{ij}$ is called sum of the matrices

$\underline{A}$ u. $\underline{B}$: $\underline{A} + \underline{B}$

analogue $\underline{A} - \underline{B}$

Example!

Mulitplication of a real number with a matrix:

$\mathbf{k \in R, \quad k \cdot \underline{A} := (k \cdot a_{ij})}$

Example!

Multiplication of a row vector with a column vector (also scalar product or innerproduct):

Row vector · column vector,

Convention: $\underline{a}$ shall always be a column vector;
a row vector shall be written as $\underline{a}^T$.

Definition: The real number (scalar)

$$\mathbf{z = a_1 \cdot b_1 + ... + a_n \cdot b_n = \sum_{i=1}^{n} a_i \cdot b_i}$$

built by the two vectors

$$\underline{\mathbf{a}} = \begin{pmatrix} \mathbf{a_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{a_n} \end{pmatrix} \quad \textbf{und} \quad \underline{\mathbf{b}} = \begin{pmatrix} \mathbf{b_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{b_n} \end{pmatrix}$$

is called scalar product $\underline{a}^T \cdot \underline{b}$ of those vectors.

$$\underline{\mathbf{a}}^{\mathbf{T}} \cdot \underline{\mathbf{b}} = (\mathbf{a_1} \quad \mathbf{a_2} \quad . \quad . \quad . \quad \mathbf{a_n}) \begin{pmatrix} \mathbf{b_1} \\ \mathbf{b_2} \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{b_n} \end{pmatrix}$$

Examples:

1) $(2 \quad 0 \quad 1 \quad 6) \cdot \begin{pmatrix} 3 \\ 4 \\ -5 \\ ½ \end{pmatrix} = 2 \cdot 3 + 0 \cdot 4 + 1 \cdot (-5) + 6 \cdot ½ = 4$

2) $\underline{u} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \quad \underline{v} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \quad \underline{u}^T \cdot \underline{v} = 0$ (orthogonal)

3) We can write the equation $4x_1 + x_2 - 3x_3 = 10$

as a scalar product $\underline{a}^T \underline{x} = 10$ by using the vectors.

$\underline{a}^T = (4 \quad 1 \quad -3)$ and $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

Multiplication of matrices

Concerning the scalar product: $\underline{a}^T \cdot \underline{b}$ we have:

Type ($\underline{a}^T$) = (1, **n**), Type (b) = (**n**, 1).

Now, we consider the matrices A and B as

$$\underline{\mathbf{A}} = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & -2 & 1 & 3 \\ 4 & -1 & 0 & 0 \end{pmatrix}, \quad \underline{\mathbf{B}} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \\ 1 & 2 \\ 0 & 0 \end{pmatrix}$$

with type (A) = (3, 4), type (B) = (4, 2)

and build all possible scalar products:

Column of B / row of A	1	2
1	3	4
2	-3	4
3	2	1

Definition: Let the (m, p) matrix A and the (p,n)-matrix B (m, n, p $\in$ N) be given.
The elements of the (m, n)-matrix $\underline{\mathbf{C}} = (\mathbf{c}_{ik})_{i=1 \dots m \quad k=1, \dots n}$
are the scalarproducts of the i[th] row of A and the k[th] column of B.
The (m, n)-matrix $\underline{\mathbf{C}} = (\mathbf{c}_{ik})_{i=1 \dots m \quad k=1, \dots n}$ is called product A · B of the matrices A and B.

Remarks:

1)

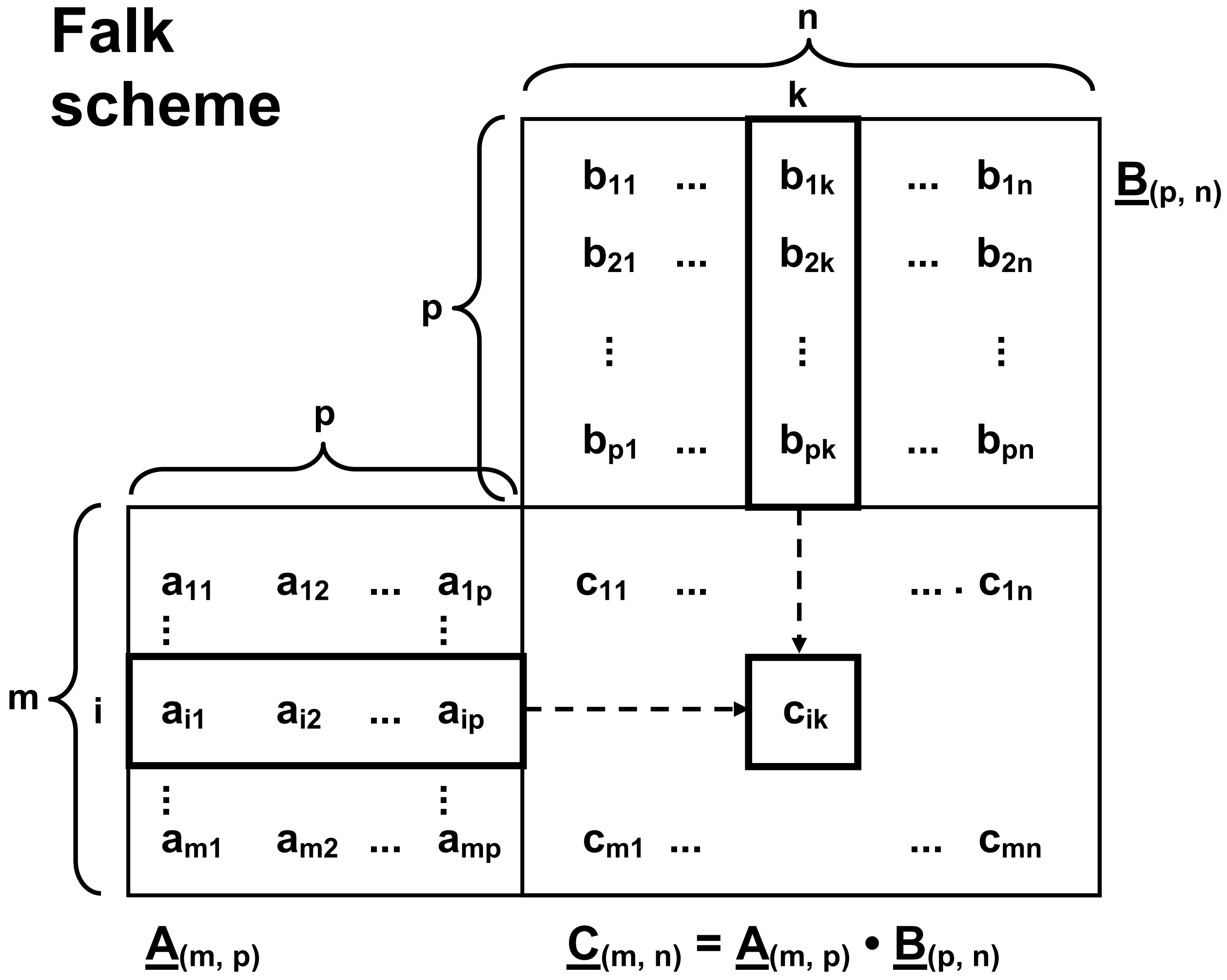
$$\underline{\mathbf{A}}\,\underline{\mathbf{B}} = \begin{pmatrix} \mathbf{a_{11}} & \cdot & \cdot & \cdot & \mathbf{a_{1p}} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \mathbf{a_{m1}} & \cdot & \cdot & \cdot & \mathbf{a_{mp}} \end{pmatrix} \begin{pmatrix} \mathbf{b_{11}} & \cdot & \cdot & \cdot & \mathbf{b_{1n}} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \mathbf{b_{p1}} & \cdot & \cdot & \cdot & \mathbf{b_{pn}} \end{pmatrix}$$

$$= \left(\sum_{r=1}^{p} \mathbf{a_{ir}} \cdot \mathbf{b_{rk}} \right)_{i=1 \dots m \quad k=1, \dots n}$$

2) To calculate the product of the matrix it is recommendable to use the Falk scheme:

				1	0
				2	-1
				1	2
				0	0
1	0	2	-1	3	4
0	-2	1	3	-3	4
4	-1	0	0	2	1

Falk scheme


n
k
p
b11 ... b1k ... b1n
b21 ... b2k ... b2n
bp1 ... bpk ... bpn
B(p, n)
p
m
i
a11 a12 ... a1p
ai1 ai2 ... aip
am1 am2 ... amp
c11 ... . c1n
cik
cm1 ... ... cmn
A(m, p)
C(m, n) = A(m, p) • B(p, n)

3) The multiplication is only possible if the number of columns of the first matrix corresponds with the number of rows of the second matrix.

4) Attention should be paid to the order of factors!

5) We have: $(\underline{A} \cdot \underline{B})^T = \underline{B}^T \cdot \underline{A}^T$

Examples:

a)

$$\begin{pmatrix} -3 & 2 & 1 \\ 0 & 4 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 2 & 1 \\ 0 & 4 & 5 \end{pmatrix}$$

$$\underline{\mathbf{A}} \cdot \underline{\underline{\mathbf{E}}} = \underline{\mathbf{A}} \; ;$$

$$\underline{\underline{\mathbf{E}}}_2 \cdot \underline{\mathbf{A}} = \underline{\mathbf{A}} \quad ; \quad \underline{\underline{\mathbf{E}}}\underline{\mathbf{x}} = \underline{\mathbf{x}}$$

b)

$$\underline{\mathbf{A}} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \quad \underline{\mathbf{B}} = \begin{pmatrix} 6 & 2 \\ -3 & -1 \end{pmatrix}$$

$$\underline{\mathbf{A}} \cdot \underline{\mathbf{B}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \underline{\mathbf{B}} \cdot \underline{\mathbf{A}} = \begin{pmatrix} 10 & 20 \\ -5 & -10 \end{pmatrix} = 5 \cdot \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix}$$

thus $\underline{A} \cdot \underline{B} \neq \underline{B} \cdot \underline{A}$

$$\underline{A} \cdot \underline{B} = \underline{0} := \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ but } \underline{A} \neq \underline{0} \text{ and } \underline{B} \neq \underline{0}$$

c)

$$\underline{A} = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 3 & 1 \end{pmatrix}, \quad \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow$$

$$\underline{A}\,\underline{x} = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 - 2x_2 + x_3 \\ 3x_2 + x_3 \end{pmatrix}$$

Let $x^0 = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \Rightarrow \underline{A}\,\underline{x}^0 = \begin{pmatrix} -4 \\ 10 \end{pmatrix}$

The matrix way of writing of a linear equation system.

$$\begin{pmatrix} 1 & -2 & 1 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -4 \\ 10 \end{pmatrix} \leftrightarrow \begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & -4 \\ & & 3x_2 & + & x_3 & = & 10 \end{array}$$

$\underline{A} \cdot \underline{x} = \underline{b}$ ($\underline{x}^0$ is a solution of the linear system of equations)

d) Economic example:

Raw materials → intermediate products → final products

3.1.6 Input / Output-analysis

American national economist Leontief (Beginnings: 1936-1941), separated the economy into sectors (branches, steps), material /value streams (in monetary units), „What do we need? What leaves the system?"

a) simple structure: raw materials/ressources → system → products
(compare to d) above)

$$\begin{pmatrix} r_1 \\ . \\ . \\ . \\ r_m \end{pmatrix} = \underline{R} \cdot \begin{pmatrix} x_1 \\ . \\ . \\ . \\ x_n \end{pmatrix}$$

b) Interwoven structure: e.g. relations between advanced series, production and services.

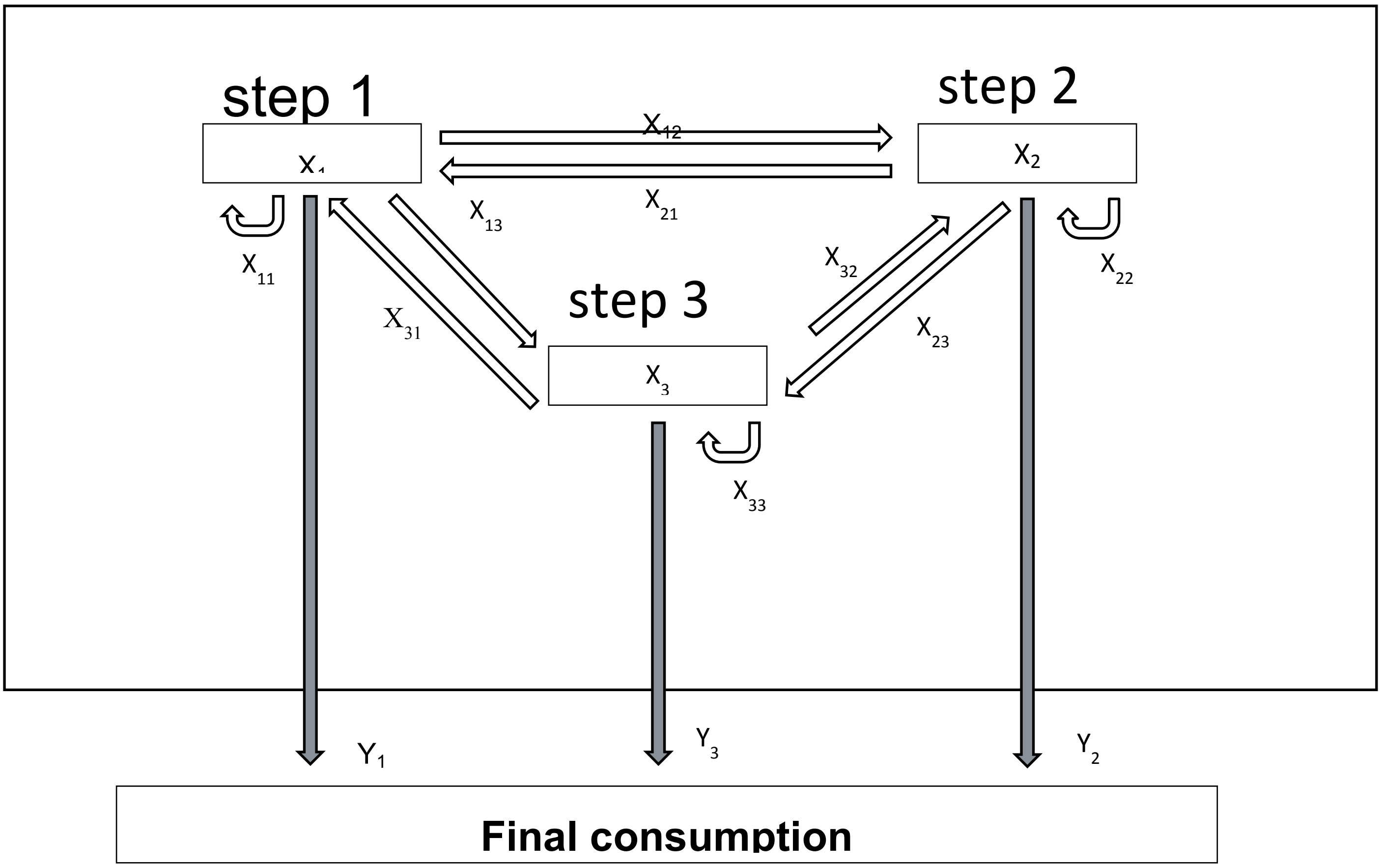
General approach n = 3:

$$\underline{X} = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$$ Material flow;

$$\underline{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$ Final consumption vector or final demand

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$ Production vector

Economical flow sheet with three branches(respectively production branches)



accounting equations:

$$x_1 = x_{11} + x_{12} + x_{13} + y_1$$
$$x_2 = x_{21} + x_{22} + x_{23} + y_2$$
$$x_3 = x_{31} + x_{32} + x_{33} + y_3$$

$$\underline{x} = \underline{X} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \underline{y}$$

$$\mathbf{a_{ij}} = \frac{\mathbf{x_{ij}}}{\mathbf{x_j}}$$

are called input-coefficient or direct-consumption-coefficent or production coefficient.

thus

$$\underline{x} = \underline{A}\,\underline{x} + \underline{y}$$

iff. $\underline{E}\,\underline{x} = \underline{A}\,\underline{x} + \underline{y}$

iff. $\underline{E}\,\underline{x} - \underline{A}\,\underline{x} = \underline{y}$

iff. $(\underline{E} - \underline{A})\,\underline{x} = \underline{y}$

Two questions:

(1) $\underline{y}$ is given, $\underline{x}$ is wanted? $\rightarrow$ later

(2) $\underline{x}$ is given, that means that we can calculate y using the matrix-multiplication.

Example:

$$\underline{x} = \begin{pmatrix} 10 \\ 20 \\ 30 \end{pmatrix}, \quad \underline{X} = \begin{pmatrix} 1 & 2 & 6 \\ 3 & 4 & 3 \\ 4 & 2 & 9 \end{pmatrix}$$ are known in a certain year.

$$\underline{A} = \left(\frac{x_{ij}}{x_j}\right) = \begin{pmatrix} \frac{1}{10} & \frac{2}{20} & \frac{6}{30} \\ \frac{3}{10} & \frac{4}{20} & \frac{3}{30} \\ \frac{4}{10} & \frac{2}{20} & \frac{9}{30} \end{pmatrix}$$

$$\underline{E} - \underline{A} = \begin{pmatrix} \frac{9}{10} & -\frac{2}{20} & -\frac{6}{30} \\ -\frac{3}{10} & \frac{16}{20} & -\frac{3}{30} \\ -\frac{4}{10} & -\frac{2}{20} & \frac{21}{30} \end{pmatrix}$$

The final consumption delivery is the result of

$$(\underline{E} - \underline{A})\ \underline{x} = \begin{pmatrix} \frac{9}{10} & -\frac{2}{20} & -\frac{6}{30} \\ -\frac{3}{10} & \frac{16}{20} & -\frac{3}{30} \\ -\frac{4}{10} & -\frac{2}{20} & \frac{21}{30} \end{pmatrix} \begin{pmatrix} 10 \\ 20 \\ 30 \end{pmatrix} = \begin{pmatrix} 1 \\ 10 \\ 15 \end{pmatrix} \begin{matrix} \textbf{branche 1} \\ \textbf{branche 2} \\ \textbf{branche 3} \end{matrix}$$

3.2 Linear combination and linear independence of vectors

3.2.1 Linear combination of vectors

Geometric interpretation of vectors

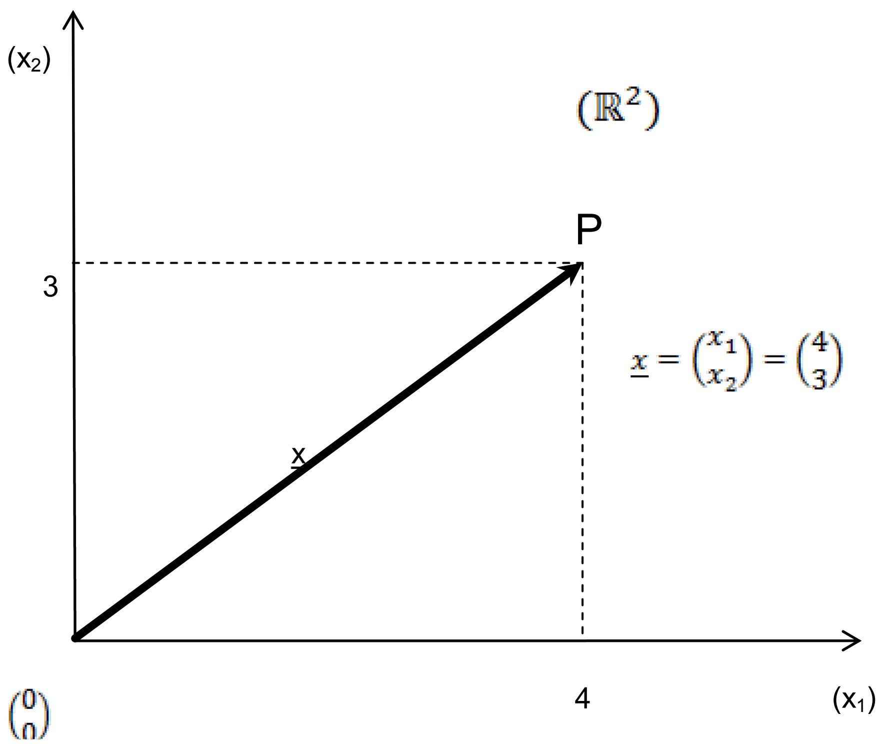
$\mathbf{R}^2$: Pairs of real numbers $\longleftrightarrow$ points in $\mathbf{x_1},\ \mathbf{x_2}$ – **plane,**

e.g.. point $\begin{pmatrix} \mathbf{4} \\ \mathbf{3} \end{pmatrix}$

$\updownarrow$

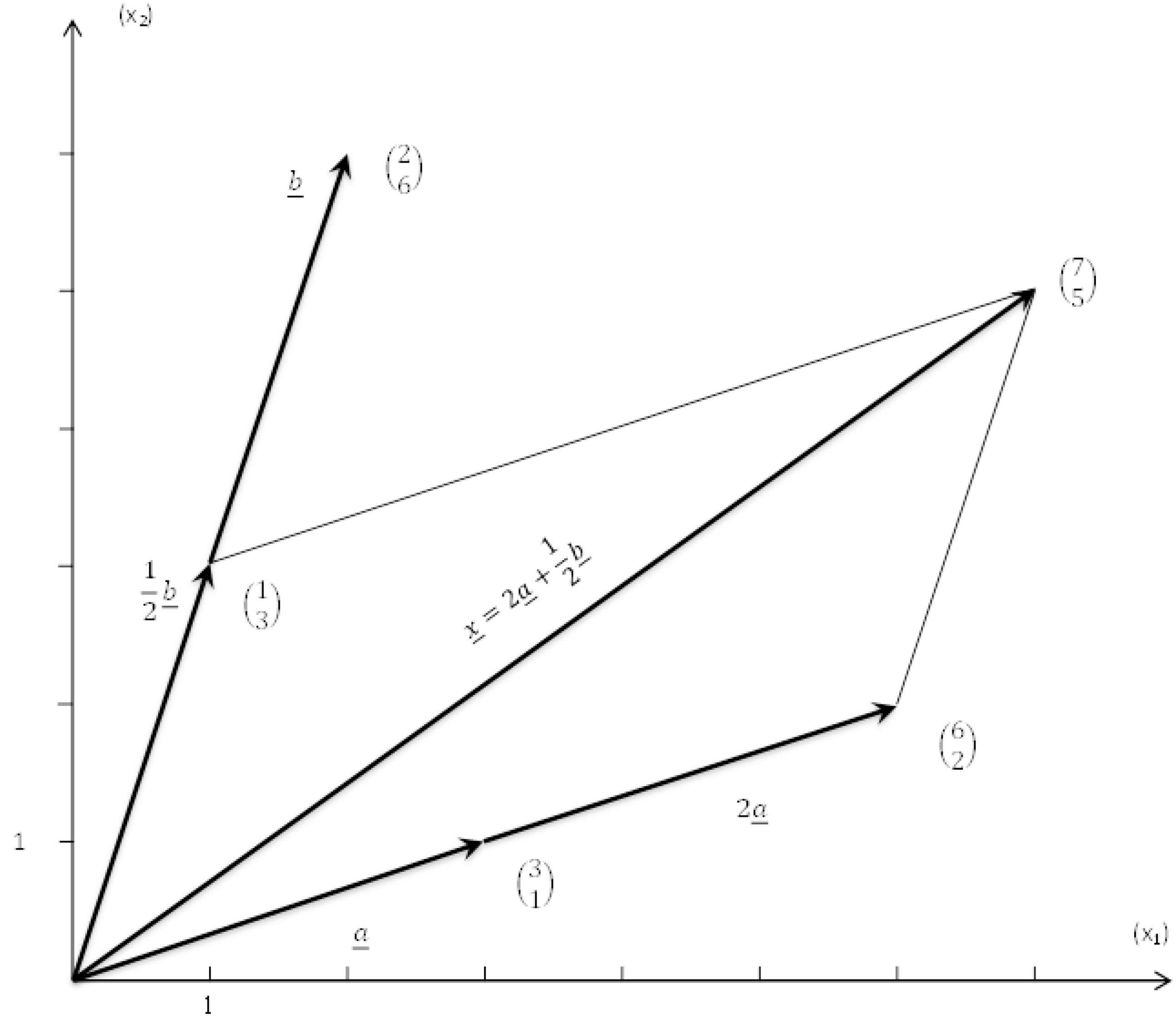
vector $\underline{\mathbf{x}} = \begin{pmatrix} \mathbf{4} \\ \mathbf{3} \end{pmatrix}$

directed distance (length, direction) also $\overrightarrow{\begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{4} \\ \mathbf{3} \end{pmatrix}}$.



$$\underbrace{\alpha \cdot \underline{\mathbf{a}}; \quad \underline{\mathbf{a}}_1 + \underline{\mathbf{a}}_2}$$

combination: e.g. $2\underline{\mathbf{a}} + \frac{1}{2}\underline{\mathbf{b}}$



R^n: n-tuples of real numbers ⟷ points in a n-dimensional space

↕

Vectors with the dimension n

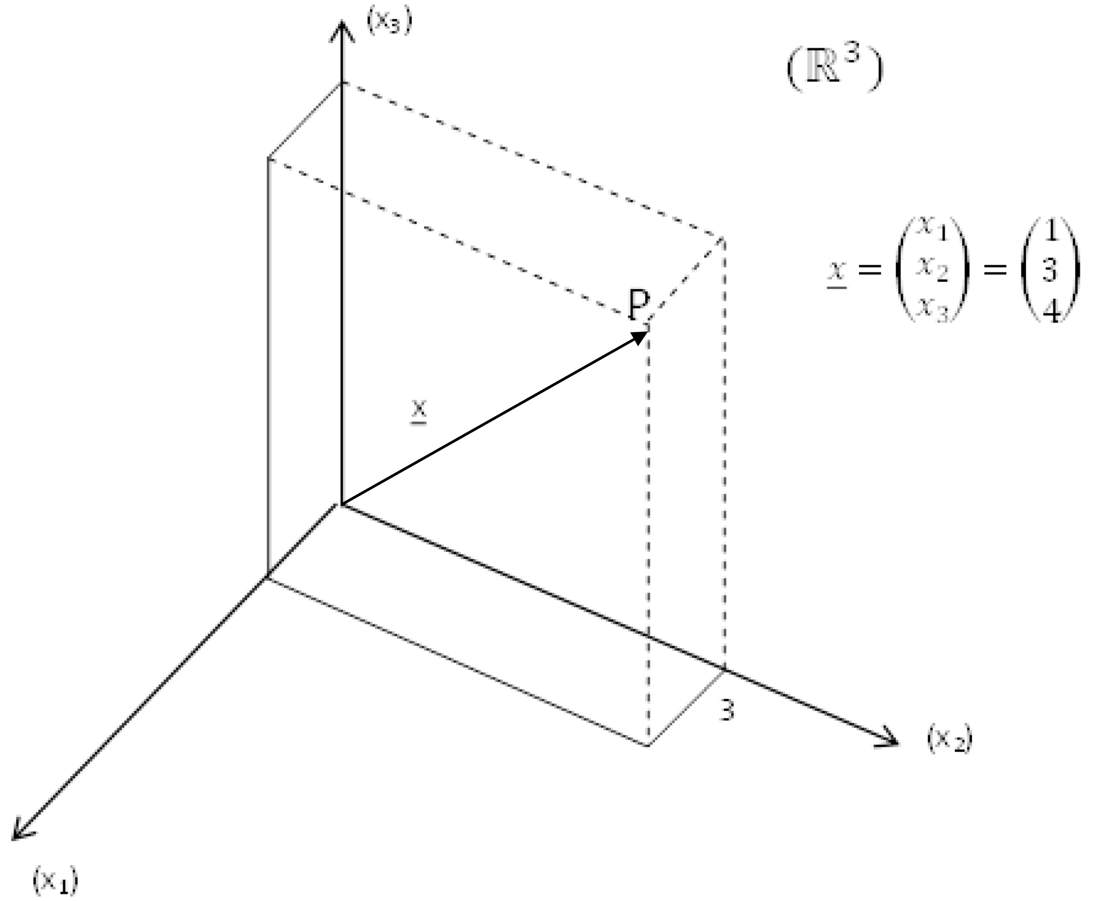
+ structure:

- Multiplication with numbers
- addition
- additional characteristics

↓

R^n: n-dimensional vector space

R^1 - line, R^2 - plane, R^3 - space



Definition:

If the vectors $\underline{\mathbf{a}}_1, \cdots, \underline{\mathbf{a}}_k \in \mathbf{R}^n$ and k real numbers $\alpha_1, \cdots, \alpha_k$ are given, the vector $\underline{\mathbf{b}} = \sum_{i=1}^{k} \alpha_i \, \underline{\mathbf{a}}_i$ is a linear combination of the vectors $\underline{\mathbf{a}}_1, \cdots, \underline{\mathbf{a}}_k$

Examples:

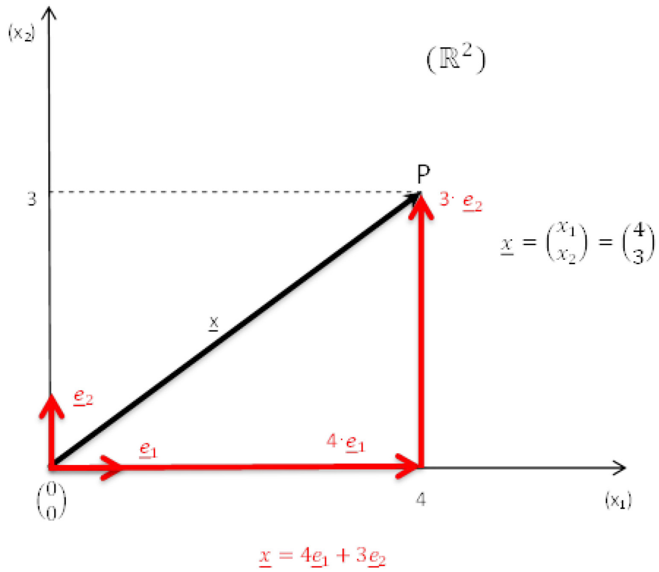
1) $\underline{\mathbf{x}} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ – **linear combination of $\underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2$;**

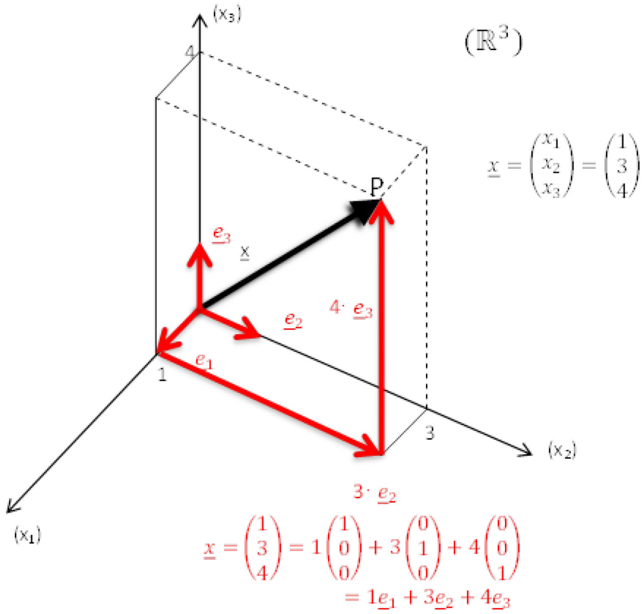
$\underline{\mathbf{x}} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$ – **linear combination of $\underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2, \underline{\mathbf{e}}_3$;**

$\underline{\mathbf{a}} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \underline{\mathbf{b}} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$ We have:

$\underline{\mathbf{x}} = 2\underline{\mathbf{a}} + 0{,}5\underline{\mathbf{b}}$ is a linear combination of a and b and

$\underline{\mathbf{x}} = 7\underline{\mathbf{e}}_1 + 5\underline{\mathbf{e}}_2$ is a linear combination of $\underline{e}_1$ and $\underline{e}_2$.





2)

Linear equation system (LES)

$$\begin{array}{rcrcrcrcr} 2x_1 & - & x_2 & + & 0{,}5\,x_3 & + & x_4 & = & 6{,}4 \\ x_1 & & & & -2\,x_3 & & & = & 0 \\ -x_1 & & -2\,x_2 & + & x_3 & & +3\,x_4 & = & 10 \end{array}$$

Equivalent way of writing with a vector:

$$\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} x_1 + \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix} x_2 + \begin{pmatrix} 0{,}5 \\ -2 \\ 1 \end{pmatrix} x_3 + \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} x_4 = \begin{pmatrix} 6{,}4 \\ 0 \\ 10 \end{pmatrix}$$

$x_1 \in R$

$\underline{a}_1 \in R^3$ $\underline{a}_2 \in R^3$ $\underline{a}_3 \in R^3$ $\underline{a}_4 \in R^3$ $\underline{b} \in R^3$

The vector $\underline{b}$ is a linear combination of $\underline{a}_1$, ..., $\underline{a}_4$.

Way of writing as a matrix:

$$\begin{pmatrix} 2 & -1 & 0{,}5 & 1 \\ 1 & 0 & -2 & 0 \\ -1 & -2 & 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 6{,}4 \\ 0 \\ 10 \end{pmatrix}$$

Coefficient matrix

$\underline{A}$

$$\underline{A} \cdot \underline{x} = \underline{b}$$

3.2.2 Linear independence of vectors

We consider systems of vectors:

(1) $\underline{\mathbf{a}}_1 = \begin{pmatrix} \mathbf{2} \\ \mathbf{1} \end{pmatrix}, \quad \underline{\mathbf{a}}_2 = \begin{pmatrix} \mathbf{1} \\ \mathbf{3} \end{pmatrix}$

Does α exist with $\underline{\mathbf{a}}_1 = \alpha \cdot \underline{\mathbf{a}}_2$? $\rightarrow$ No!

(The vectors are on different lines)!

(2) $\underline{\mathbf{a}}_1 = \begin{pmatrix} \mathbf{2} \\ \mathbf{1} \end{pmatrix}, \quad \underline{\mathbf{a}}_2 = \begin{pmatrix} \mathbf{1} \\ \mathbf{3} \end{pmatrix}, \quad \mathbf{a}_3 = \begin{pmatrix} \mathbf{3} \\ \mathbf{4} \end{pmatrix}$

Is it possible to get a vector as linear combination of the other vectors?

$\rightarrow$ Yes!

$$\underline{\mathbf{a}}_3 = \mathbf{1} \cdot \underline{\mathbf{a}}_1 + \mathbf{1} \cdot \underline{\mathbf{a}}_2$$

$\downarrow\uparrow$

$$\underline{\mathbf{a}}_1 + \underline{\mathbf{a}}_2 - \mathbf{1} \cdot \underline{\mathbf{a}}_3 = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} = \underline{\mathbf{0}}$$

Definition:

The vectors $\mathbf{R}^n : \underline{\mathbf{a}}_1, \ldots, \underline{\mathbf{a}}_k \in \mathbf{R}^n$ are called **linear independent** if

$\underline{\mathbf{0}} = \alpha_1 \cdot \underline{\mathbf{a}}_1 + \alpha_2 \, \underline{\mathbf{a}}_2 + \ldots + \alpha_k \cdot \underline{\mathbf{a}}_k$ always implies that

$\alpha_1 = \alpha_2 = \ldots = \alpha_k = 0$,

otherwise they are called linear dependent.

Remarks:

1) The definition includes a condition which says that it is only possible to express the nullvector as linear combination of the vectors $\underline{\mathbf{a}}_1, ..., \underline{\mathbf{a}}_k$ in a trivial way.

2) $\underline{\mathbf{a}}_1, ..., \underline{\mathbf{a}}_k$ are linear dependent , if and only if there is at least one vector $\alpha_i, \quad i \in \{1, ..., k\} \quad with \quad \alpha_i \neq 0$,which can be expressed as linear combination of the other vectors.

3) A subset of linear independent vectors is also linear independent.

Example: (1), (2) (see above)

Let k > n, then it holds: k-vectors out of R^n (more than n vectors) are always linear dependent.

The unit vectors

$$\underline{\mathbf{e}}_1 = \begin{pmatrix} 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix}, \underline{\mathbf{e}}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix}, \cdots, \underline{\mathbf{e}}_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \cdots, \quad \underline{\mathbf{e}}_n = \begin{pmatrix} 0 \\ . \\ . \\ . \\ 0 \\ 1 \end{pmatrix}$$

i-th coordinate

of R^n are linear independent.

3.3 Basis of a vector space, elementary transformation of a basis

Examples for n linear independent vectors in R^n:

$$\mathbf{R}^2: \quad \underline{\mathbf{a}}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \ \underline{\mathbf{a}}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad ; \quad \{\underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2\}$$

$\mathbf{R}^n$: $\underline{\mathbf{e}}_1, \ldots, \underline{\mathbf{e}}_n$ standard basis

Definition: A set of n linear independent vectors in the n-dimensional vector space R^n is called **basis** of the vector space R^n

- The set of the following vectors $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ **and** $\begin{pmatrix} 6 \\ 3 \end{pmatrix}$ is not a basis.
- $\mathbf{B} = \{\underline{\mathbf{b}}_1, \ldots, \underline{\mathbf{b}}_n\}$ basis $\Rightarrow$ create all the linear combinations $\Rightarrow \mathbf{R}^n$
- If $\underline{\mathbf{c}} \in \mathbf{R}^n \Rightarrow$ we have a definite linear combination concerning the basis:

  $\underline{\mathbf{c}} = \mathbf{c}_1 \cdot \underline{\mathbf{b}}_1 + \ldots + \mathbf{c}_n \, \underline{\mathbf{b}}_n$

  $\mathbf{c}_1, \ldots, \mathbf{c}_n$ are called coordinates concerning this basis.

Transition from one basis to another !?

Example:

$\mathbf{B}_1 = \{\underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2\}$ → **$\underline{a}_2$ in the basis in exchange for $\underline{e}_1$** → $\mathbf{B}_2 = \{\underline{\mathbf{a}}_2, \underline{\mathbf{e}}_2\}$ → **$\underline{a}_1$ in the basis in exchange for $\underline{e}_2$** → $\mathbf{B}_3 = \{\underline{\mathbf{a}}_1, \underline{\mathbf{a}}_2\}$

Definition: We have one basis $\underline{\mathbf{b}}_1, \ldots, \underline{\mathbf{b}}_n$ out of R^n and another vector $\underline{\mathbf{a}} \in \mathbf{R}^n$.

The transition to a new basis is called elementary transformation of a basis, if it is possible to interchange the vector $\underline{\mathbf{a}}$ and the basis vector $\underline{\mathbf{b}}_i, \quad \mathbf{i} \in \{1, \ldots, \mathbf{n}\}$, so that $\underline{\mathbf{b}}_1, \underline{\mathbf{b}}_2, \ldots, \underline{\mathbf{b}}_{i-1}, \underline{\mathbf{a}}, \underline{\mathbf{b}}_{i+1}, \ldots, \underline{\mathbf{b}}_n$ is a new basis of R^n.

Elementary transformation of a basis: Calculation tableau

		$\underline{a}_1$	$\underline{a}_2$ ⇓
⇐	$\underline{e}_1$	2	1
	$\underline{e}_2$	1	3

		$\underline{a}_1$ ⇓	$\underline{e}_1$
	$\underline{a}_2$	2	1
⇐	$\underline{e}_2$	-5	-3

$$\textbf{I} \quad \underline{a}_1 = 2\underline{e}_1 + \underline{e}_2$$
$$\textbf{II} \quad \underline{a}_2 = \underline{e}_1 + 3\underline{e}_2$$

$$\textbf{II} \rightarrow \quad \underline{e}_1 = \underline{a}_2 - 3\underline{e}_2$$

$$\textbf{in I:} \quad \underline{a}_1 = 2\underline{a}_2 - 6\underline{e}_2 + \underline{e}_2$$
$$= 2\underline{a}_2 - 5\underline{e}_2$$

$$\textbf{I} \quad \underline{a}_1 = 2\underline{a}_2 - 5\underline{e}_2$$
$$\textbf{II} \quad \underline{e}_1 = \underline{a}_2 - 3\underline{e}_2$$

$$\textbf{I} \rightarrow \quad 5\underline{e}_2 = 2\underline{a}_2 - \underline{a}_1 \rightarrow \quad \underline{e}_2 = \frac{2}{5}\underline{a}_2 - \frac{1}{5}\underline{a}_1$$

$$\textbf{in II:} \quad \underline{e}_1 = \underline{a}_2 - 3\left(\frac{2}{5}\underline{a}_2 - \frac{1}{5}\underline{a}_1\right)$$
$$= -\frac{1}{5}\underline{a}_2 + \frac{3}{5}\underline{a}_1$$

	$\underline{e}_2$	$\underline{e}_1$
$\underline{a}_2$	$\frac{2}{5}$	$-\frac{1}{5}$
$\underline{a}_1$	$-\frac{1}{5}$	$\frac{3}{5}$

→ arrange

	$\underline{e}_1$	$\underline{e}_2$
$\underline{a}_1$	$\frac{3}{5}$	$-\frac{1}{5}$
$\underline{a}_2$	$-\frac{1}{5}$	$\frac{2}{5}$

(Algorithmic) rules R

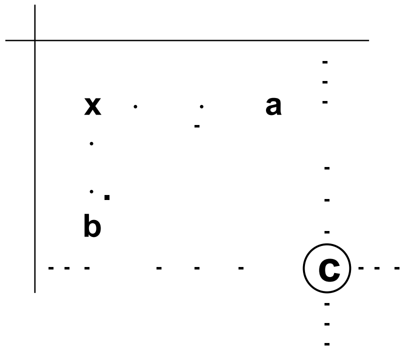
How do we get the elements (numbers, coordinates) of the new table out of the elements of the old tableau?

(1) **Central or pivot element** (c) shall be transformed to: $\frac{1}{c}$,

(2) The other elements of the **pivot row**
shall be multiplied by: $\frac{1}{c}$,

(3) The other elements of the **pivot column**
shall be multiplied by: $-\frac{1}{c}$,

(4) To get the remaining elements you have to use the **cross** rule:



x shall be transformed to : $x - \frac{a \cdot b}{c}$

Elementary transformation of a basis and linear dependence of vectors

Are the following vectors $\underline{a}_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, $\underline{a}_2 = \begin{pmatrix} -10 \\ 2 \\ 3 \end{pmatrix}$, $\underline{a}_3 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

linear dependent or linear independent?

	$\underline{a}_1$	$\underline{a}_2$	$\underline{a}_3$
$\underline{e}_1$	0	–10	–2
$\underline{e}_2$	1	2	0
$\underline{e}_3$	–1	3	(1)

	$\underline{a}_1$	$\underline{a}_2$	$\underline{e}_3$
$\underline{e}_1$	–2	–4	2
$\underline{e}_2$	(1)	2	0
$\underline{a}_3$	–1	3	1

	$\underline{e}_2$	$\underline{a}_2$	$\underline{e}_3$
$\underline{e}_1$	2	[0]	2
$\underline{a}_1$	1	2	0
$\underline{a}_3$	1	5	1

The central element shall differ from 0.

⇨ „Final tableau“

Evaluation of column $\underline{a}_2$:

$\underline{a}_2 = 2\underline{a}_1 + 5\underline{a}_3$

Theorem:

Let's suppose we have a basis in $\mathbf{R}^n$ and r additional vectors $\underline{\mathbf{a}}_1, \underline{\mathbf{a}}_2, ..., \underline{\mathbf{a}}_r \in \mathbf{R}^n$.

The vectors $\underline{\mathbf{a}}_1, \underline{\mathbf{a}}_2, ..., \underline{\mathbf{a}}_r$ are linear independent if it is possible to transfer them all together in the basis by using the elementary transformation of a basis.

The rank of a matrix

In general: The maximum number of linear independent rows of a matrix and the maximum number of linear independent columns of a matrix are identic.

Definition: The maximum number of linear independent columns (respectively rows) of matrix $\underline{A}$ is called **rank** ($\rho(\underline{A})$ also $r(\underline{A})$) of matrix $\underline{A}$.

Example:

$$\text{Let } \underline{\mathbf{A}} = \begin{pmatrix} 1 & 0 & 2 \\ 3 & -5 & 1 \\ 1 & -1 & 1 \\ -2 & 1 & -3 \end{pmatrix}, \quad \rho(\underline{\mathbf{A}}) = ?$$

Determine the rank by using elementary transformation of a basis:

	$\underline{a}_1$	$\underline{a}_2$	$\underline{a}_3$
$\underline{e}_1$	1	0	2
$\underline{e}_2$	3	–5	1
$\underline{e}_3$	1	–1	1
$\underline{e}_4$	–2	1	–3

	$\underline{e}_1$	$\underline{a}_2$	$\underline{a}_3$
$\underline{a}_1$	1	0	2
$\underline{e}_2$	–3	–5	–5
$\underline{e}_3$	–1	–1	–1
$\underline{e}_4$	2	1	1

	$\underline{e}_1$	$\underline{e}_4$	$\underline{a}_3$
$\underline{a}_1$	1	0	2
$\underline{e}_2$	7	5	0
$\underline{e}_3$	1	1	0
$\underline{a}_2$	2	1	1

⇒ $\rho(\underline{A}) = 2$

Definition: A (n, n)-matrix $\underline{A}$ is called regular, if $\rho(\underline{A}) = n$.
If $\rho(\underline{A}) < n$, the matrix is called singular.

Another designation for “regular”: „matrix with full rank“.

3.4 Linear equation systems

Definition: A system,

$$
\begin{array}{lllllll}
a_{11}x_1 & + a_{12}x_2 & + & \dots & + a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + a_{22}x_2 & + & \dots & + a_{2n}x_n & = & b_2 \\
\vdots & & & & \vdots & & \\
a_{m1}x_1 & + a_{m2}x_2 & + & \dots & + a_{mn}x_n & = & b_m
\end{array}
$$

with the constant parameters

a_{ij} for $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$,

b_i for $i = 1, 2, ..., m$

and the variables

x_j for $j = 1, 2, ..., n$ is called

Linear equation system with m equations and n variables.

If all b_i for $i = 1, 2, ..., m$ are equal to 0,then the linear equation system is called **homogeneous**, otherwise it is called **inhomogeneous**.

Vector presentation:

$$\underline{a}_1 x_1 + \underline{a}_2 x_2 + \ldots + \underline{a}_n x_n = \underline{b}$$

with $\underline{a}_i = \begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{pmatrix}$, für $i = 1, 2, \ldots, n$ und $\underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$.

Matrix presentation:

$$\underline{A}\, \underline{x} = \underline{b}$$

with $\underline{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$

The term of a solution of a linear equation system:

Definition: a vector $x = \hat{x}$ of fixed values, which satisfies the condition $\underline{A}\, \hat{\underline{x}} = \underline{b}$ (which transfers it in an identity), is called a **solution** of the linear equation system $\underline{A}\, \underline{x} = \underline{b}$.

Example:

$$\begin{aligned} 2x_1 + x_2 - x_3 &= 5 \\ -3x_1 + \frac{1}{2}x_3 &= 10 \\ x_2 - x_3 &= 0 \end{aligned}$$

We solve the linear equation system by using elementary transformations of a basis:

	$\underline{a}_1$	$\underline{a}_2$ ↓	$\underline{a}_3$	$\underline{b}$
← $\underline{e}_1$	2	(1)	–1	5
$\underline{e}_2$	–3	0	1/2	10
$\underline{e}_3$	0	1	–1	0

	$\underline{a}_1$	$\underline{e}_1$	$\underline{a}_3$ ↓	$\underline{b}$
$\underline{a}_2$	2	1	–1	5
← $\underline{e}_2$	–3	0	(1/2)	10
$\underline{e}_3$	–2	–1	0	–5

	$\underline{a}_1$ ↓	$\underline{e}_1$	$\underline{e}_2$	$\underline{b}$
$\underline{a}_2$	–4	1	2	25
$\underline{a}_3$	–6	0	2	20
← $\underline{e}_3$	(–2)	–1	0	–5

	$\underline{e}_3$	$\underline{e}_1$	$\underline{e}_2$	$\underline{b}$
$\underline{a}_2$	–2	3	2	35
$\underline{a}_3$	–3	3	2	35
$\underline{a}_1$	–1/2	1/2	0	5/2

$$\underline{\mathbf{b}} = \underline{\mathbf{a}}_1 \cdot \frac{5}{2} + \underline{\mathbf{a}}_2 \cdot 35 + \underline{\mathbf{a}}_3 \cdot 35$$

x_1 x_2 x_3

Solution: $$\underline{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5/2 \\ 35 \\ 35 \end{pmatrix}$$

(Proof!)

We had three equations and three variables in the example above. There was exactly one solution, which is not always the case.

Quadratic equation system: m=n	<u>A</u> singular ↓ it is possible ∄(no solution) / many solutions
Overdetermined equation system: m>n	it is possible mostly: ∄(no solution) / ∃! (exactly one solution) / many solutions
Underdetermined equation system: m<n	it is possible mostly: many solutions / ∄(no solution)

Two questions:

a) Solvability: Does the linear equation system have a solution?

b) Uniqueness: How many solutions does the linear equation system have?

Theorem: A linear equation system $\underline{A}\ \underline{x} = \underline{b}$ is solvable,if and only if $\rho(\underline{A}) = \rho(\underline{A},\underline{b})$

$\underline{A},\underline{b}$ is called extended coefficient matrix.

In the example above it is $\rho(\underline{A}) = 3 = \rho(\underline{A},\underline{b})$.

Theorem: Given the linear equation systems $\underline{A}\ \underline{x} = \underline{b}$ with n variables ($\underline{x} \in R^n$).
The linear equation system has exactly one solution if and only if $\rho(\underline{A}) = n$

If, in contrast, it holds $\rho(\underline{A}) = \rho(\underline{A},\underline{b}) = r < n$, then $f = n - r$ variables are free to choose, viz. we have an infinite number of solutions.

f is the degree of freedom of the linear equation system

Examples:

1) task above: $\rho(\underline{A}) = 3 = n$

2) $3x_1 - x_2 + 2x_3 + x_4 = 5$

The linear equation system is composed of one equation with four variables.

$\rho(\underline{A}) = \rho(\underline{A},\underline{b}) = 1, \quad n = 4, \quad f = 3;$

$x_4 = 5 - 3x_1 + x_2 - 2x_3, \quad x_1, x_2, x_3 \in R$ arbitrary

3) $2x_1 + 2x_2 + 2x_3 = 1$

$x_1 - 4x_2 + 3x_3 - 2x_4 = -1$

The linear equation system is composed of two equations with four variables.

Solution of the linear equation system using elementary transformations of a basis:

	$\underline{a}_1$	$\underline{a}_2$	$\underline{a}_3$	$\underline{a}_4$	$\underline{b}$
$\underline{e}_1$	2	2	2	0	1
$\underline{e}_2$	1	-4	3	-2	-1

	$\underline{e}_1$	$\underline{a}_2$	$\underline{a}_3$	$\underline{a}_4$	$\underline{b}$
$\underline{a}_1$	1/2	1	1	0	1/2
$\underline{e}_2$	–1/2	-5	2	-2	-3/2

	$\underline{e}_1$	$\underline{a}_2$	$\underline{a}_3$	$\underline{e}_2$	$\underline{b}$
$\underline{a}_1$	1/2	1	1	0	1/2
$\underline{a}_4$	1/4	5/2	-1	-1/2	3/4

General solution:

$$\begin{pmatrix} x_1 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 3/4 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 5/2 & -1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ x_3 \end{pmatrix}, \qquad x_2, x_3 \in \mathbf{R} \quad \text{arbitrary}$$

Basis variable (BV) (x_1, x_4) Non-basic variable (NBV) (x_2, x_3)

We get special solutions by fixing the arbitrary variables:

a) $x_2 = 1$, $x_3 = 1$ ⇒ $x_1 = -\frac{3}{2}$, $x_4 = -\frac{3}{4}$ thus $\underline{x} = \begin{pmatrix} -3/2 \\ 1 \\ 1 \\ -3/4 \end{pmatrix}$

b) $x_2 = 0$, $x_3 = 1$ ⇒ $x_1 = -\frac{1}{2}$, $x_4 = \frac{7}{4}$ thus $\underline{x}$ $\begin{pmatrix} -1/2 \\ 0 \\ 1 \\ 7/4 \end{pmatrix}$

3.5 The inverse of a quadratic matrix

Definition: If for a quadratic matrix A there exists a quadratic matrix $\underline{A}^{-1}$ with $\underline{A}^{-1} \cdot \underline{A} = \underline{A} \cdot \underline{A}^{-1} = \underline{E}$, then $\underline{A}^{-1}$ is called the **inverse** of $\underline{A}$.

A regular matrix has exactly one inverse. If an inverse exists, then the matrix is regular.

$\underline{E}^{-1} = \underline{E}$

$(\underline{A}^{-1})^{-1} = \underline{A}$

Calculation of the inverse by using elementary transformations of a basis.

(There are other methods we will not deal with here: Gaussian elimination, Cramer's Rule):

	$\underline{a}_1$	...	$\underline{a}_n$
$\underline{e}_1$			
⋮		$\underline{A}$	
$\underline{e}_n$			

After ETBs and arranging the vectors →

	$\underline{e}_1$	...	$\underline{e}_n$
$\underline{a}_1$			
⋮		$\underline{A}^{-1}$	
$\underline{a}_n$			

Examples:

1) $\underline{A} = \begin{pmatrix} 4 & 2 \\ 5 & 3 \end{pmatrix}$

	$\underline{a}_1$	$\underline{a}_2$
$\underline{e}_1$	4	(2)
$\underline{e}_2$	5	3

	$\underline{a}_1$	$\underline{e}_1$
$\underline{a}_2$	2	1/2
$\underline{e}_2$	(-1)	-3/2

	$\underline{e}_2$	$\underline{e}_1$
$\underline{a}_2$	2	-5/2
$\underline{a}_1$	-1	3/2

ar-range →

	$\underline{e}_1$	$\underline{e}_2$
$\underline{a}_1$	3/2	-1
$\underline{a}_2$	-5/2	2

Thus $\underline{A}^{-1} = \begin{pmatrix} 3/2 & -1 \\ -5/2 & 2 \end{pmatrix}$ (Proof!)

2) $\underline{A} = \begin{pmatrix} 2 & 1 & -1 \\ -3 & 0 & 1/2 \\ 0 & 1 & -1 \end{pmatrix}$

Compare with the coefficient matrix of the first equation system of chapter 3.4

Arrange $\Rightarrow$ $\underline{A}^{-1} = \begin{pmatrix} 1/2 & 0 & -1/2 \\ 3 & 2 & -2 \\ 3 & 2 & -3 \end{pmatrix}$ (Prove $\underline{A} \cdot \underline{A}^{-1}$ = ?)

Matrix equations with the inverse

- Linear equation system $\underline{A}\,\underline{x} = \underline{b}$

(1) Let's suppose $\underline{A}$ is regular, $\rho(\underline{A}) = n$, $\underline{x} \in R^n$:

$\underline{A}^{-1}$ exists and

$\underline{A}\,\underline{x} = \underline{b}$ is equivalent to

$\underbrace{\underline{A}^{-1}\,\underline{A}}_{\underline{E}}\,\underline{x} = \underline{A}^{-1}\,b$, that means $\underline{x} = \underline{A}^{-1}\,\underline{b}$ is the unique solution of the linear equation system

(2) Now let's suppose $\underline{A}$ is a (r, n)-matrix with $\rho(\underline{A}) = \rho(\underline{A},\underline{b}) = r < n$:

we decompose $\underline{A}$ and $\underline{x}$ (by interchanging the corresponding columns and variables) in

$\underline{A} = (\underline{B}, \underline{N})$, so that $\underline{B}$ is a regular matrix, and

$\underline{x}$ in $\underline{x}_B$ and $\underline{x}_N$.

Out of $\underline{A}\,\underline{x} = \underline{b}$ respectively

$\underline{B}\,\underline{x}_B + \underline{N}\,\underline{x}_N = \underline{b}$ results via multiplication by $\underline{B}^{-1}$ from the left

$\underline{B}^{-1}\,\underline{B}\,\underline{x}_B + \underline{B}^{-1}\,\underline{N}\,\underline{x}_N = \underline{B}^{-1}\,\underline{b}$ respectively

$\underline{x}_B = \underline{B}^{-1}\,\underline{b} - \underline{B}^{-1}\,\underline{N}\,\underline{x}_N$.

The vector of the basic variables $\underline{x}_B$ is now represented by the vector of the non-basic variables $\underline{x}_N$, which is equivalent to the general solution of the linear equation system.

- Input / Output-analysis (Part 2)
  $(\underline{E} - \underline{A})\, \underline{x} = \underline{\mathbf{y}}$

(a) $\underline{x}$ is given, $\underline{y}$ is requested: matrix multiplication $\underline{y} = (\underline{E} - \underline{A})\, \underline{x}$

(b) $\underline{y}$ is given, $\underline{x}$ is requested: $\underline{x} = (\underline{E} - \underline{A})^{-1}\, \underline{y}$,
that means we need to calculate the inverse of $\underline{E} - \underline{A}$.

3.6. Linear optimization, simplex algorithm

Linear optimization (or linear programming): subject of applied mathematics with a well-founded theory,

The state Brandenburg would like to support the sustainable land use. For this purpose the state will use two programs: the exensification programm (EX) and the organic farming support program (OFS). Concerning the sustainable land use, the extensification program has a target contribution of 3 and the organic farming support program has a target contribution of 5. Six million euros will be available for both programs.
The implementation of the programs creates administrative expenses: The implementation of EX will need one civil servant per one million euro, the implementation of OFS will need two civil servant per 1 million euro, whereas seven civil servants will be available for both programs.
In addition, the implementation of the programs will need additional farm workload: The implementation of EX will need additional farm workload of three workers per one million euro, the implementation of OFS will need nine workers per one million euro, whereas the total additional workload should be lower than 27 workers.
To what extent EX and OFS should be supported?

Variables: x1 Mio. € for OFS
x2 Mio. € for EX.

The mathematic modelling produces the following linear optimization problem:

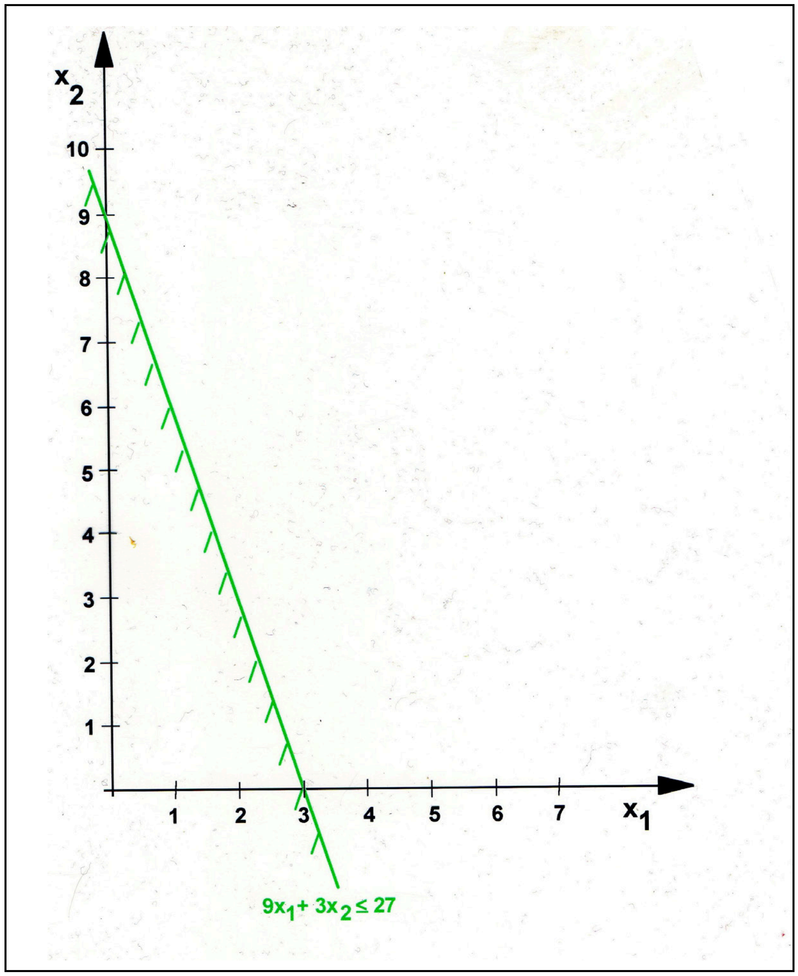
$$\max\left\{ 5x_1 + 3x_2 \;\middle|\; \begin{array}{c} 9x_1 + 3x_2 \le 27 \\ 2x_1 + x_2 \le 7 \\ x_1 + x_2 \le 6 \end{array} ,\, x_1 \ge 0,\; x_2 \ge 0 \right\}$$

with the objective function(OF) and the constraints;

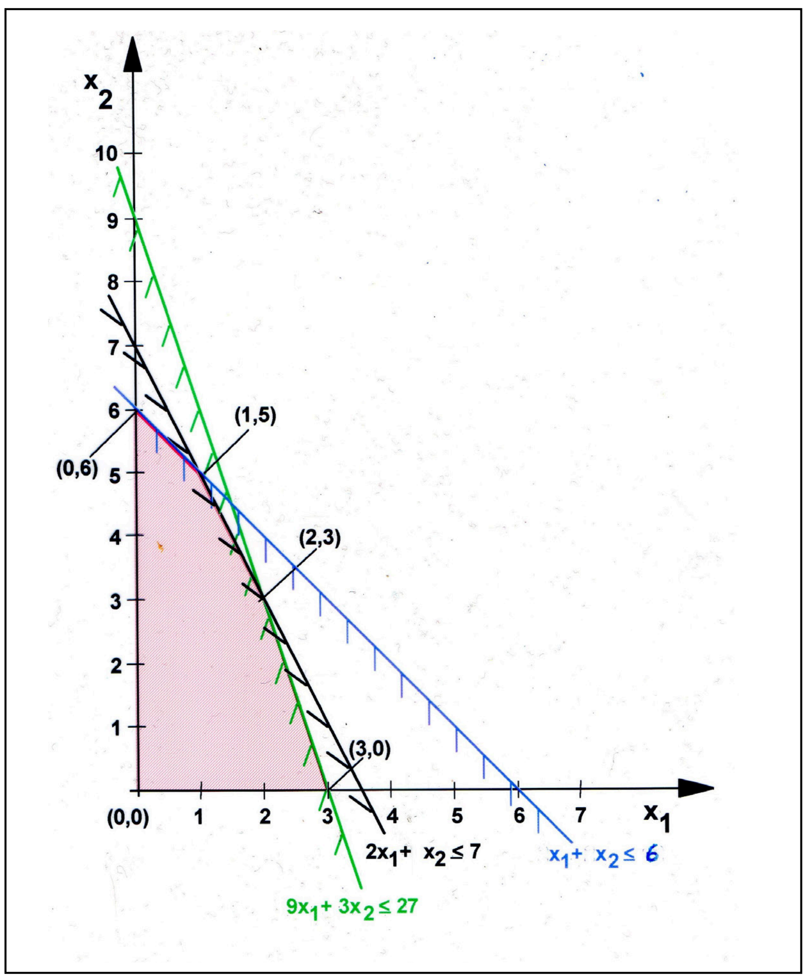
- considering two variables it is possible to solve the problem and to represent it geometrically
- but the geometric solution is inexact and only applicable to a limited extent (2 variables)

Geometric solution

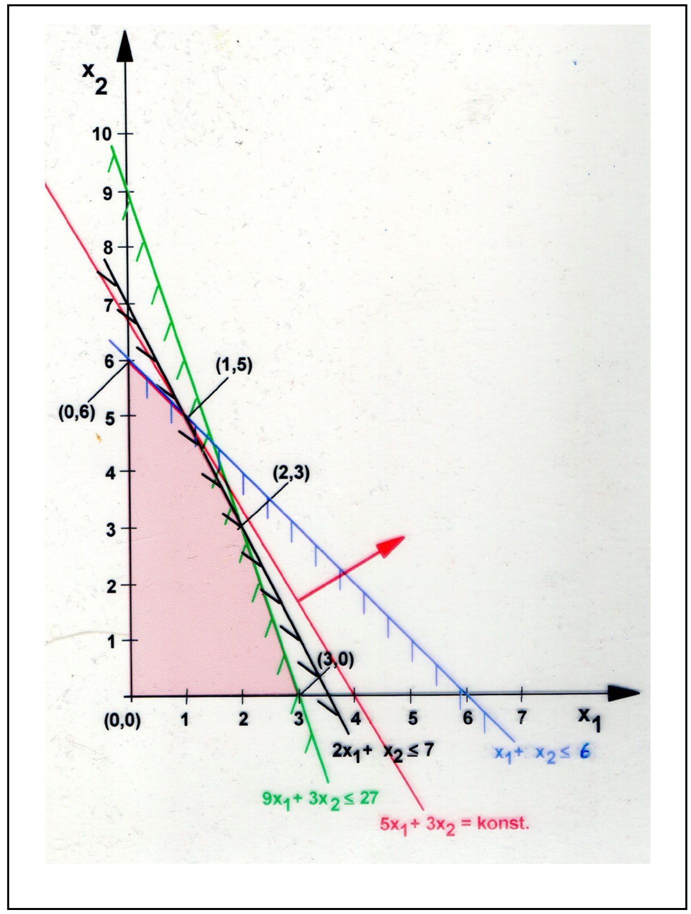
First halfplane



Set of feasible solutions



Optimum solution in point (1, 5) as point of contact (for a parallel shift) of the objective function and the set of feasible solution



Arithmetic solution

Simplex algorithm (G. B. Dantzig, 1948/49),
For that we transfer three inequalities by introducing slack variables

x_3, x_4, x_5 (all ≥ 0) in equations: $$9x_1 + 3x_2 + x_3 = 27$$ $$2x_1 + x_2 + x_4 = 7$$ $$x_1 + x_2 + x_5 = 6$$, $OF: 5x_1 + 3x_2 + 0x_3 + 0x_4 + 0x_5$,

Resolving into slack variables.: $$x_3 = 27 - (9x_1 + 3x_2)$$ $$x_4 = 7 - (2x_1 + x_2)$$ $$x_5 = 6 - (x_1 + x_2)$$

Basic variables (BV): x_3, x_4, x_5

Non basic variables (NBV): x_1, x_2

creates the following Simplex tableau:

			5	3
			x_1	x_2
0	x_3	27	9	3
0	x_4	7	2	1
0	x_5	6	1	1
		0	-5	-3

BV: x_3 = 27, x_4 = 7, x_5 = 6

NBV: x_1 = 0, x_2 = 0

We get a new simplex tableau,this is similar to ETB, but we need to choose the pivot element so that the objective function improves.

Rules:

<u>Pivot column:</u> min {-5, -3 } = -5

(see last row: *Characteristic row*)

If the definite value is ≥ 0 , the related vector is a optimal solution.

<u>Pivot row:</u>

$$\min\left\{\frac{27}{9},\frac{7}{2},\frac{6}{1}\right\}=\frac{27}{9}$$

By minimizing we consider only those fractions having a positive denominator. If every values of the pivot column are negative, there is no optimal solution.

			0	3
			x_3	x_2
5	x_1	3	1/9	1/3
0	x_4	1	-2/9	1/3
0	x_5	3	-1/9	2/3
		15	5/9	-4/3

BV: x_1 = 3, x_4 = 1, x_5 = 3

NBV: x_3 = 0, x_2 = 0

PC: $\min\left\{\frac{5}{9},\frac{-4}{3}\right\}=\frac{-4}{3}$

PR: $\min\left\{\frac{3}{\frac{1}{3}},\frac{1}{\frac{1}{3}},\frac{3}{\frac{2}{3}}\right\}=\frac{1}{\frac{1}{3}}$

			0	0
			x_3	x_4
5	x_1	2	1/3	-1
3	x_2	3	-2/3	3
0	x_5	1	1/3	-2
		19	-1/3	4

BV: x_1 = 2, x_2 = 3, x_5 = 1

NBV: x_3 = 0, x_4 = 0

PC: $\min\left\{\frac{-1}{3},4\right\}=\frac{-1}{3}$

PR: $\min\left\{\frac{2}{\frac{1}{3}},\frac{1}{\frac{1}{3}}\right\}=\frac{1}{\frac{1}{3}}$

			0	0
			x_5	x_4
5	x_1	1	-1	1
3	x_2	5	2	-1
0	x_3	3	3	-6
		20	1	2

BV: x_1 = 1, x_2 = 5, x_3 = 3 (not required workers) NBV: x_5 = 0, x_4 = 0

$$\min\{1,2\}=1\geq 0$$

$\Rightarrow$ We got an optimal solution!

Linear optimization problem:

Maximizing or minimizing a linear function under the constraints of a system of linear equations and inequalities and non negative variables.

max $\{\underline{c}^T \underline{x} | \underline{x} \epsilon B\}$

$$B = \left\{ \underline{x} \in R^n \left| \begin{array}{lll} \sum_{j=1}^{n} a_{ij} x_j = b_i & , & i \in E \\ \sum_{j=1}^{n} a_{ij} x_j \leq b_i & , & i \in LE \quad , \quad \underline{x} \geq \underline{0} \\ \sum_{j=1}^{n} a_{ij} x_j \geq b_i & , & i \in GE \end{array} \right. \right\}$$

E, LE , GE are index sets

B is a convex polyhedron: intersection of a finite number of closed half-spaces.

We have:

min $\{\underline{c}^T \underline{x} | \underline{x} \epsilon B\}$ = - max$\{-\underline{c}^T \underline{x} | \underline{x} \epsilon B\}$, that means a linear minimizing problem can be reduced to a maximizing problem.

(1) $B \neq \emptyset$ there is

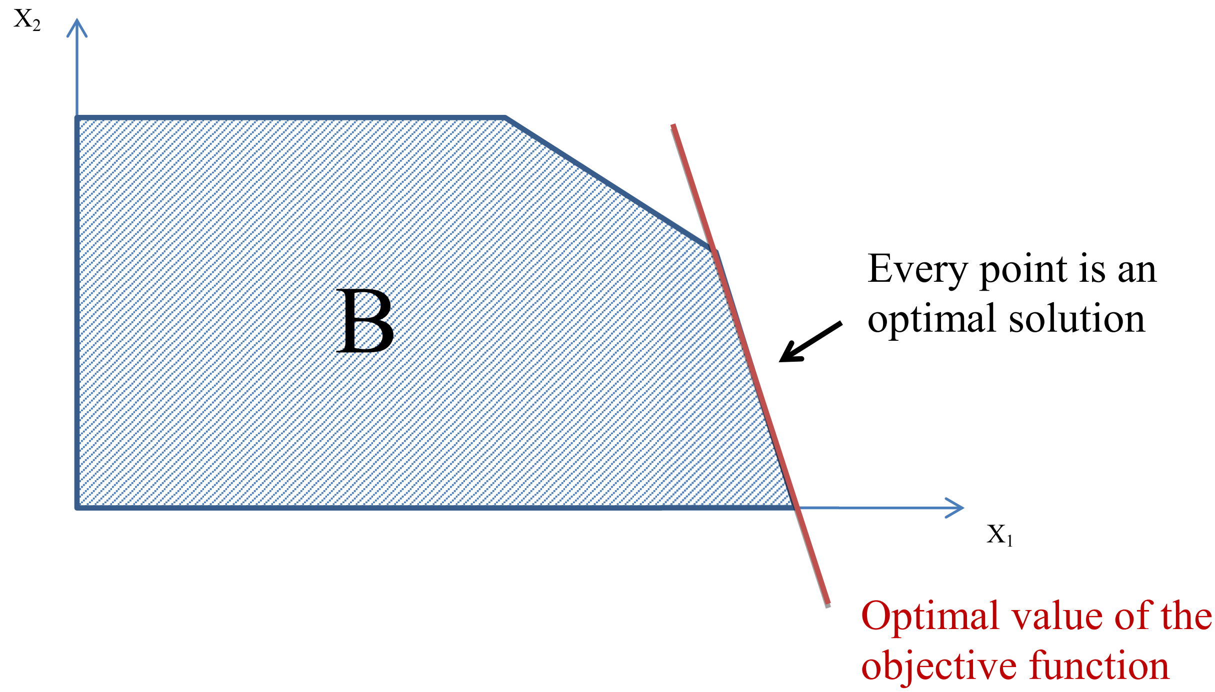
a) exactly one optimal solution (a corner point of the convex polyhedron B),see above ,

b) an infinite number of optimal solutions (an edge or a side of the convex polyhedron B)

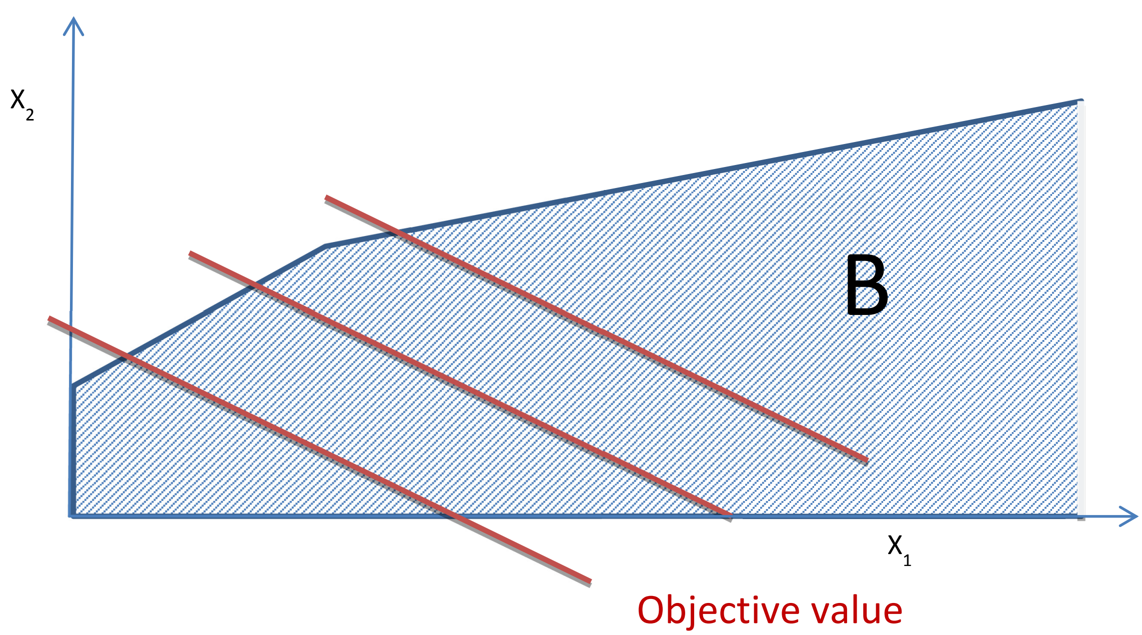
c) no optimal solution.

(2) $B = \emptyset$

b)


X2
B
Every point is an optimal solution
X1
Optimal value of the objective function

c)


X2
B
X1
Objective value

(2) $B = \emptyset$ For example: three half spaces which do not overlap all together.